Using Auctions to Allocate Transportation Requests for Demand Responsive Transit Systems

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Abstract

Demand responsive transit (DRT) systems provide flexible transportation services where individual passengers request door-to-door rides by specifying desired pick-up and drop-off locations and times. Multiple shuttles service these requests in shared-ride mode without fixed routes and schedules. In this report, we define the online cost-sharing problem in DRT systems and describe typical cost-sharing mechanisms, focusing on proportional and incremental cost sharing and some of their shortcomings in the online setting, where knowledge of future arrivals of passengers is missing. We then determine properties of cost-sharing mechanisms that we believe make DRT systems attractive to both providers and passengers, namely online fairness, immediate response, budget balance and ex-post incentive compatibility. We propose a novel cost-sharing mechanism, called Proportional Online Cost Sharing (POCS), and use an analogy involving water in containers to motivate it. POCS provides passengers with upper bounds on their fares immediately after their arrival, allowing the passengers to accept or decline. Thus, passengers have no uncertainty about whether they can be serviced or how high their fares will be at most, while the DRT systems reduce their uncertainty about passengers dropping out and can thus prepare better. Yet, they still retain some flexibility to optimize the routes and schedules after future arrivals. The sum of the fares of all passengers always equals the operating cost. Thus, no profit is made and no subsidies are required. POCS provides incentives for passengers to arrive as early as possible since the fares of passengers per mile of requested travel are never higher than those of passengers who arrive after them (and, informally, also since the likelihood of transportation capacity being available tends to decrease over time). Thus, the DRT systems have more time to prepare and

might also be able to offer subsequent passengers lower fares due to synergies with the early ride requests, which might allow them to service more passengers.

Disclosure

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Introduction

Demand responsive transit (DRT) systems provide flexible transportation services where individual passengers request door-to-door rides by specifying desired pick-up and drop-off locations and times. Multiple shuttles (or vans or small busses) service these requests in shared-ride mode without fixed routes and schedules. DRT services are more flexible and convenient for passengers than busses since they do not operate on fixed routes and schedules, yet cheaper than taxis due to the higher utilization of transportation capacity. In the United States, DRT services are commonly used to service the transportation needs of disabled and elderly citizens and have experienced rapid growth (4, 18), for example, in form of dial-a-ride paratransit services mandated under the Americans with Disabilities Act. Furthermore, the National Transit Summaries and Trends report for 2008 states that the average operating cost per passenger trip is \$30.0 for DRT systems but only \$3.3 for busses, the average operating cost per passenger mile is \$3.4 for DRT systems but only \$0.8 for buses, and the revenue from fares covers less then 10% of the operating costs of DRT systems. Hence, it is important to identify opportunities for reductions in cost and improvements in efficiency (7, 18), especially if one wants to expand DRT services to provide a transit option for urban populations in general.

Two important research issues in the context of DRT systems are how to determine the routes and schedules of the shuttles (including how to assign passengers to shuttles) in the presence of conflicting objectives, such as maximizing the number of serviced passengers, minimizing the operating cost or minimizing the passenger inconvenience, and how much to charge the passengers. The first (optimization) problem has received considerable attention in the literature. The second (cost-sharing) problem, on the other hand, has been neglected in the literature, which might be due to DRT providers being highly subsidized and most passengers thus enjoying transportation services at affordable fares, typically determined by flat rates within service zones that do not cover the operating cost. Without subsidies, the fares would substantially increase and passengers would then be more concerned about how the operating cost is shared among them in a fair manner.

The optimization problem is often solved as vehicle routing problem in a centralized way (2, 5, 6, 10, 18, 20, 24). We, on the other hand, are interested in solving it with versions of sequential single-item auctions in a decentralized way by exploiting that shuttles can easily be equipped with general-purpose computers. Auctions are promising distributed methods for teams of agents to assign and re-assign tasks among themselves in time-constrained, dynamic and only partially known situations (13, 19). In general, auction-based coordination systems can be efficient both in communication, as agents communicate only essential summary information, and in computation, as agents compute their bids in parallel. Addressing the optimization problem with auctions requires one to extend ideas from task-allocation problems in the context of multi-agent routing to ones in the context of DRT systems.

During our project, we discovered quickly that the optimization and cost-sharing problems are highly interrelated since the routes and schedules of the shuttles determine the operating cost that needs to be shared, while the cost-sharing mechanism imposes constraints on the routes and schedules that need to be optimized, for example, because the fares of passengers should not exceed the fare quotes. How passengers should share the operating cost is a non-trivial problem for the following reasons: Passengers do not arrive (that is, submit their ride requests) at the same time but should be provided with incentives to arrive as early as possible since, this way, the DRT systems have more time to prepare and might also be able to offer subsequent passengers lower fares due to synergies with the early ride requests, which might allow them to service more passengers. Passengers have different pick-up and drop-off locations and times and thus cause different amounts of inconvenience to the other passengers, which should be reflected in the fares. Passengers should be quoted fares immediately after their arrival because, this way, they have no uncertainty about whether they can be serviced or how high their fares will be at most and the DRT systems reduce their uncertainty about passengers dropping out and can thus prepare better. This requires DRT systems to make instantaneous and irreversible decisions despite having no knowledge of future arrivals (3).

We therefore adapted the original project plan and focused our effort on the cost-sharing mechanism. In this report, we define the online cost-sharing problem in DRT systems and describe typical cost-sharing mechanisms, focusing on proportional and incremental cost sharing and some of their shortcomings in the online setting. We then determine properties of cost-sharing mechanisms that we believe make DRT systems attractive to both providers and passengers, namely online fairness, immediate response, budget balance and ex-post incentive compatibility. We propose a novel cost-sharing mechanism, called Proportional Online Cost Sharing (POCS), and use an analogy involving water in containers to motivate it. Finally, we show that POCS satisfies our properties.

2

Online Cost Sharing

In this chapter, we define the online cost-sharing problem for demand responsive transit (DRT) systems, provide an example and discuss existing cost-sharing mechanisms and some of their shortcomings in the online setting, and finally derive a list of desirable properties for online cost-sharing mechanisms for DRT systems.

2.1 Problem Definition

DRT systems provide flexible transportation services where individual passengers request door-to-door rides. Multiple shuttles service these requests without fixed routes and schedules. Passengers share shuttles. For example, after a passenger has been picked up and before it is dropped off, other passengers can be picked up and dropped off, resulting in a longer ride for the passenger. Passengers need to pay a share of the operating cost. Passengers arrive (that is, submit their ride requests) one after the other by specifying their desired pick-up and drop-off locations and time. The arrival time of a passenger is the time when it submits its ride request. (In case the passenger decides to delay its arrival, we distinguish its truthful arrival time, which is the earliest possible arrival time, from its actual, perhaps delayed, arrival time.) We assume, for simplification, that all passengers arrive before the shuttles start to service passengers. We also assume, without loss of generality, that exactly one passenger arrives at each time $k = 1, \ldots, t$, namely that passenger $\pi(k)$ arrives at time k under arrival order π, where an arrival order is a function that maps arrival times to passengers.

Definition 1. *For all times k with $1 \leq k$ and all arrival orders π, the alpha value $\alpha_{\pi(k)}$ of passenger $\pi(k)$ quantifies the demand of its request, that is, how much of the transportation resources it requests. We assume that it is positive and independent of the arrival time of the passenger.*

These assumptions are, for example, satisfied for the shortest point-to-point travel distance from the pick-up to the drop-off location of a passenger, which is the quantity that we use in this report for its alpha value.

Definition 2. *For all times t with $1 \leq t$ and all arrival orders π, the total cost $totalcost^t_\pi$ at time t under arrival order π is the operating cost required to service passengers $\pi(1), \ldots, \pi(t)$. We define $totalcost^0_\pi := 0$ and assume that 1) the total cost is non-decreasing over time, that is, for all times t and t' with $t \leq t'$ and all arrival orders π, $totalcost^t_\pi \leq totalcost^{t'}_\pi$; and 2) the total cost at time t is independent of the arrival order of passengers $\pi(1), \ldots, \pi(t)$, that is, for all times t with $1 \leq t$ and all arrival orders π and π' with $\{\pi(1), \ldots, \pi(t)\} = \{\pi'(1), \ldots, \pi'(t)\}$, $totalcost^t_\pi = totalcost^t_{\pi'}$.*

These assumptions are, for example, satisfied for the minimal operating cost, which is the quantity that we use in this report for the total cost. The DRT system can accommodate advanced features, such as operating times and capacities of shuttles and time constraints of passengers, as long as it can determine total costs that satisfy the assumptions. (The assumptions are typically not satisfied if passengers can arrive after the shuttles have started to service passengers since the shuttle locations influence the total cost.) We assume, for simplification, that the DRT system can easily calculate the total cost at any given time although this could be NP-hard and thus time-consuming, such as for the minimal operating cost.

Definition 3. *For all times k with $1 \leq k$ and all arrival orders π, the marginal cost $mc_{\pi(k)}$ of passenger $\pi(k)$ under arrival order π is the increase in total cost due to its arrival, that is, $mc_{\pi(k)} := totalcost^k_\pi - totalcost^{k-1}_\pi$.*

Definition 4. *For all times k and t with $1 \leq k \leq t$ and all arrival orders π, the shared cost $cost^t_{\pi(k)}$ of passenger $\pi(k)$ at time t under arrival order π is its share of the total cost at time t.*

The DRT system provides a (myopic) fare quote to a passenger immediately after its arrival. The fare quoted to passenger $\pi(k)$ after its arrival at time k is $cost^k_{\pi(k)}$. (A fare quote of infinity means that the passenger cannot get serviced.) The passenger

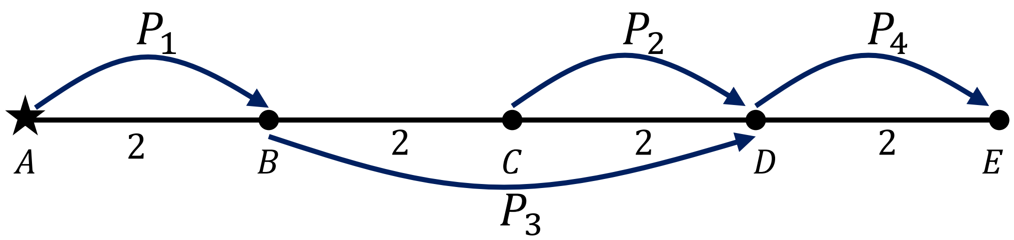


Figure 2.1: DRT Example 1

		$k=1$ $\pi(k)=P_1$	$k=2$ $\pi(k)=P_2$	$k=3$ $\pi(k)=P_3$	$k=4$ $\pi(k)=P_4$
Alpha Value:	$\alpha_{\pi(k)}$	2	2	4	2
Total Cost:	$totalcost^k_\pi$	20	60	60	80
Marginal Cost:	$mc_{\pi(k)}$	20	40	0	20

Table 2.1: DRT Values

then accepts or declines. If it declines, the DRT system simply pretends that it never arrived, which explains why we assume, without loss of generality, that all passengers accept. If it accepts, it will get serviced. Its fare is $cost^t_{\pi(k)}$ (which does not necessarily equal the fare quote) if the shuttles start to service passengers after the arrival of passenger $\pi(t)$.

2.2 Demand Responsive Transit Example

We use the DRT example in Figure 2.1 to illustrate typical cost-sharing mechanisms. There is one shuttle that can transport up to four passengers and starts at the star. The shuttle incurs an operating cost of 10 for each unit of distance traveled and does not need to return to its initial location. There are four passengers with arrival order $\pi(1) = P_1$, $\pi(2) = P_2$, $\pi(3) = P_3$ and $\pi(4) = P_4$. For example, Passenger P_3 requests a ride from location B to location D, as shown in Figure 2.1. All passengers accept all fare quotes. Table 2.1 shows the alpha value of each passenger, the total cost after the arrival of each passenger and the marginal cost of each passenger. For example, the alpha value of Passenger P_3 is the point-to-point travel distance from its pick-up location B to its drop-off location D. Thus, $\alpha_{\pi(3)} = 4$. The total cost at time 3, after the arrival of Passenger P_3, is 10 times the minimal travel distance of the shuttle required

	$k=1$ $\pi(k)=P_1$	$k=2$ $\pi(k)=P_2$	$k=3$ $\pi(k)=P_3$	$k=4$ $\pi(k)=P_4$
$t=1$	20			
$t=2$	30	30		
$t=3$	15	15	30	
$t=4$	16	16	32	16

Table 2.2: Shared Costs under Proportional Cost Sharing: $cost^t_{\pi(k)}$

to service Passengers P_1, P_2 and P_3. Thus, $totalcost^3_\pi = 60$ since the shuttle has to drive from location A (to pick up Passenger P_1) via location B (to drop off Passenger P_1 and pick up Passenger P_3) and location C (to pick up Passenger P_2) to location D (to drop off Passengers P_2 and P_3). The marginal cost of Passenger P_3 is the increase in total cost due to its arrival. Thus, $mc_{\pi(3)} = totalcost^3_\pi - totalcost^2_\pi = 60 - 60 = 0$ since the total cost remains 60.

2.3 Typical Cost-Sharing Mechanisms

Online cost-sharing mechanisms determine the shared costs in the online setting, where knowledge of future arrivals of passengers is missing. We present typical cost-sharing mechanisms and some of their shortcomings in the online setting, using the DRT example in Section 2.2.

2.3.1 Proportional Cost Sharing

One commonly used cost-sharing mechanism is *proportional cost sharing* (26, 27), where the total cost is distributed among all passengers proportionally to their alpha values, which reflects that passengers with higher demands should contribute more toward the total cost. Consequently, for all times k and t with $1 \leq k \leq t$ and all arrival orders π, the shared cost of passenger $\pi(k)$ at time t under arrival order π is

$$cost^t_{\pi(k)} := totalcost^t_\pi \frac{\alpha_{\pi(k)}}{\sum_{j=1}^{t} \alpha_{\pi(j)}}.$$

Instead of distributing the total (operating) cost among all passengers, one could also distribute the operating cost of each shuttle among all passengers serviced by that

	Truthful Arrival				Delayed Arrival			
	$k=1$	$k=2$	$k=3$	$k=4$	$k=1$	$k=2$	$k=3$	$k=4$
	$\pi(k)=P_1$	$\pi(k)=P_2$	$\pi(k)=P_3$	$\pi(k)=P_4$	$\pi(k)=P_2$	$\pi(k)=P_1$	$\pi(k)=P_3$	$\pi(k)=P_4$
$t=1$	20				60			
$t=2$	20	40			60	0		
$t=3$	20	40	0		60	0	0	
$t=4$	20	40	0	20	60	0	0	20

Table 2.3: Shared Costs under Incremental Cost Sharing: $cost^t_{\pi(k)}$

shuttle, which results in identical properties for the DRT example in Section 2.2 since there is only one shuttle in the DRT example.

Table 2.2 shows the shared costs for the DRT example. For example, the total cost at time 3 is 60. It is distributed among all passengers that have arrived at time 3, namely Passengers P_1, P_2 and P_3, proportionally to their alpha values, namely 2, 2 and 4, respectively. Consequently, the shared cost of Passenger P_3 at time 3 and thus the fare quoted to Passenger P_3 after its arrival is $cost^3_{\pi(3)} = 30$. Similarly, the total cost at time 4 is 80. It is distributed among all passengers that have arrived at time 4, namely Passengers P_1, P_2, P_3 and P_4, proportionally to their alpha values, namely 2, 2, 4 and 2, respectively. Consequently, the shared cost of Passenger P_3 at time 4 and thus its fare is $cost^4_{\pi(3)} = 32$, implying that its fare is higher than its fare quote. This is undesirable because Passenger P_3 might agree with the fare quote but not the higher fare, meaning that it will have to drop out shortly before receiving its ride and then needs to search for a last-minute alternative to using the DRT system, which might be pricy and is not guaranteed to exist. This is undesirable, which is why we suggest that a fare quote should be an upper bound on the fare (immediate response property). We also suggest that the upper bound should be reasonably low since passengers might otherwise look for alternatives to using the DRT system after receiving a fare quote that they do not agree with, commit to one and then drop out unnecessarily. Obtaining reasonably low upper bounds can be difficult since the DRT system has no knowledge of future arrivals.

2.3.2 Incremental Cost Sharing

Another commonly used cost-sharing mechanism is *incremental cost sharing* (16), where the shared cost of each passenger is its marginal cost, which is the increase in total cost due to its arrival. Consequently, for all times k and t with $1 \leq k \leq t$ and all arrival orders π, the shared cost of passenger $\pi(k)$ at time t under arrival order π is

$$cost^t_{\pi(k)} := mc_{\pi(k)}.$$

Table 2.3 (left) shows the shared costs for the DRT example in Section 2.2. For example, the marginal cost of Passenger P_3 is 0. Consequently, the shared cost of Passenger P_3 from its arrival at time 3 on is 0, and thus both its fare quote and fare are 0 as well. In general, incremental cost sharing satisfies the immediate response property since the marginal costs are independent of time. The fares of Passengers P_1, P_2 P_3 and P_4 are 20, 40, 0 and 20, respectively. Thus, Passenger P_3 is a free rider, which is undesirable in general and especially in the context of the DRT example since Passenger P_3 has the highest demand, which should be reflected in the fares. Proportional cost sharing does not suffer from this problem. For the discussion below, notice that the fare per alpha value of Passenger P_1 is 10 and the one of Passenger P_3 is 0 even though Passenger P_1 arrives before Passenger P_3.

Table 2.3 (right) shows the shared costs for the DRT example in Section 2.2 if Passenger P_1 delays its arrival and the passengers arrive in order P_2, P_1, P_3 and P_4. Now, the shared cost of Passenger P_1 from its arrival at time 2 on is 0, and thus both its fare quote and fare are 0 as well. Thus, Passenger P_1 can reduce its fare from 20 to 0 by strategically delaying its arrival. This is undesirable because DRT systems have more time to prepare the earlier passengers arrive, which is why we suggest to provide incentives for passengers to arrive truthfully (that is, as early as possible), for example by ensuring that the fares per alpha value of passengers are never higher than those of passengers who arrive after them (online fairness property). Incremental cost sharing does not satisfy this property as shown above. In general, we suggest to ensure that the best strategy of every passenger is to arrive truthfully because it cannot decrease its fare by delaying its arrival (incentive compatibility property).

2.3.3 Other Cost-Sharing Mechanisms

There has been a considerable amount of research on designing cost-sharing mechanisms in the fields of cooperative game theory and multi-agent systems, mostly focusing on the offline setting, see (9, 11, 21) for related work. Two cost-sharing mechanisms have received a fair amount of attention in the mechanism design literature in addition to proportional and incremental cost sharing, namely the *value mechanism* (22) and the *serial mechanism* (15). However, these mechanisms for the offline setting encounter similar problems as proportional cost sharing in the online setting. Online cost-sharing mechanisms have been studied in (3) but without considering fairness.

Online fair division problems for cake cutting (25) and resource allocation (12) deal with agents that arrive one after the other, just like the online cost-sharing problem for DRT systems, but the available amount of cake and resources of those problems do not depend on the requests of the agents, while the total cost of online cost-sharing problems for DRT systems depends on the ride requests..

2.4 Desirable Properties

None of the cost-sharing mechanisms discussed so far are well-suited for the on-line setting. Based on their shortcomings, we derive a list of desirable properties for online cost-sharing mechanism. Our primary objective is to design online cost-sharing mechanisms that provide incentives for passengers to arrive truthfully while satisfying basic properties of cost-sharing mechanism in general, such as fairness and budget balance.

- **Online Fairness:** The shared costs per alpha value of passengers are never higher than those of passengers who arrive after them, that is, for all times k_1, k_2 and t with $1 \leq k_1 \leq k_2 \leq t$ and all arrival orders π,

$$\frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \leq \frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}}.$$

- **Immediate Response:** Passengers are provided at their arrival with (ideally low) upper bounds on their shared costs at any future time, that is, for all times k, t_1 and t_2 with $1 \leq k \leq t_1 \leq t_2$ and all arrival orders π,

$$cost^{t_1}_{\pi(k)} \geq cost^{t_2}_{\pi(k)}.$$

- **Budget Balance:** The total cost equals the sum of the shared costs of all passengers, that is, for all times $1 \leq t$ and all arrival orders π,

$$\sum_{j=1}^{t} cost^t_{\pi(j)} = totalcost^t_{\pi}.$$

- **Ex-Post Incentive Compatibility (EPIC):**[1] The best strategy of every passenger is to arrive truthfully provided that all other passengers arrive truthfully as well and do not change whether they accept or decline their fare quotes because it then cannot decrease its shared cost by delaying its arrival, that is, for all times k_1, k_2 and t with $1 \leq k_1 < k_2 \leq t$ and all arrival orders π and π' with

$$\pi'(k) = \begin{cases} \pi(k+1) & \text{if } k_1 \leq k < k_2 \\ \pi(k_1) & \text{if } k = k_2 \\ \pi(k) & \text{otherwise,} \end{cases}$$

$$cost^t_{\pi(k_1)} \leq cost^t_{\pi'(k_2)}.$$

The online fairness and ex-post incentive compatibility (EPIC) properties are similar but one does not imply the other. Basically, they provide incentives for passengers to arrive truthfully. Thus, the DRT systems have more time to prepare and might also be able to offer subsequent passengers lower fares due to synergies with the early ride requests, which might allow them to service more passengers. The online fairness property is also meant to ensure that passengers consider the fares to be fair (since they understand that passengers need to be provided with incentives to arrive early since their early arrivals often affect other passengers positively, yet often result in high fare quotes for them since potential synergies with later ride requests are not yet known at their arrival), although this still needs to be tested empirically. The immediate response property enables DRT systems to provide fare quotes in form of upper bounds

[1] We would like the ex-post incentive compatibility property ideally to state that the best strategy of every passenger is to arrive truthfully because it cannot decrease its shared cost by delaying its arrival. However, we impose two conditions in this report that we hope to be able to relax in the future. The first condition is that all other passengers arrive truthfully, which, for example, rules out collusion of several passengers. In general, the literature on online mechanism design (17) distinguishes two types of incentive compatibility, namely *dominant-strategy incentive compatibility* and *ex-post incentive compatibility* (EPIC). Dominant-strategy incentive compatibility does not require the first condition, while ex-post incentive compatibility (EPIC) does. Dominant-strategy incentive compatibility is difficult to achieve in the online setting, which is why we (and other researchers) are content with imposing the first condition in this report. The second condition is that none of the other passengers change whether they accept or decline their fare quotes, even though, for example, the delayed arrival of a passenger could cause the fare quotes of subsequent passengers to increase, which might make them drop out. The arrival orders with and without the delayed arrival are then difficult to relate, which is why we are content with imposing the second condition in this report.

on the fares to passengers immediately after their arrival despite missing knowledge of future arrivals. Thus, passengers have no uncertainty about whether they can be serviced or how high their fares will be at most, while the DRT systems reduce their uncertainty about passengers dropping out and can thus prepare better. Yet, they still retain some flexibility to optimize the routes and schedules after future arrivals. The budget balance property guarantees that the sum of the fares of all passengers always equals the operating cost. Thus, no profit is made and no subsidies are required.

We stated sufficient rather than necessary conditions for the properties. For example, the immediate response property could be weakened to state that passengers are provided at their arrival with (ideally low) upper bounds on their shared costs after the arrival of the last passenger since this implies that their fare quotes are upper bounds on their fares. Similarly, the budget balance property could be weakened to state that the total cost equals the sum of the shared costs of all passengers after the arrival of the last passenger. Requiring the properties to hold at any time rather than only after the arrival of the last passenger simplifies the development of online cost-sharing mechanism since they do not know in advance at which time the last passenger arrives.

3

Proportional Online Cost Sharing

Several online cost-sharing mechanisms might satisfy the properties listed in Section 2.4. In this chapter, we describe a novel online cost-sharing mechanism, called Proportional Online Cost Sharing (POCS), which satisfies the properties and is thus a very first step toward addressing some of the problems raised by missing knowledge of future arrivals. The idea behind POCS is the following: POCS partitions passengers into coalitions, where coalitions contain all passengers that arrive within given time intervals (rather than, for example, all passengers are served by the same shuttle). Initially, each newly arriving passenger forms its own coalition. However, passengers can choose to form coalitions with passengers that arrive directly after them to decrease their shared costs per alpha value, which implies the online fairness, immediate response, and ex-post incentive compatibility (EPIC) properties. For example, the immediate response property holds because passengers add other passengers to their coalitions only when this decreases their shared costs per alpha value and thus also their shared costs (since the alpha values are positive). We prove in Chapter 4 that POCS indeed satisfies all properties listed in Section 2.4.

3.1 Calculation of Shared Costs

We start by defining the coalition cost per alpha value to be able to describe formally how POCS calculates the shared costs.

Definition 5. *For all times k_1, k_2 and t with $k_1 \leq k_2 \leq t$ and all arrival orders π, the coalition cost per alpha value of passengers $\pi(k_1), \ldots, \pi(k_2)$ at time t under arrival* *order π is*

$$ccpa_{\pi(k_1,k_2)} := \frac{\sum_{j=k_1}^{k_2} mc_{\pi(j)}}{\sum_{j=k_1}^{k_2} \alpha_{\pi(j)}}.$$

We now describe how POCS calculates the shared costs.

Definition 6. *For all times k and t with $k \leq t$ and all arrival orders π, the shared cost of passenger $\pi(k)$ at time t under arrival order π is*

$$cost^t_{\pi(k)} := \alpha_{\pi(k)} \min_{k \leq j \leq t} \max_{1 \leq i \leq j} ccpa_{\pi(i,j)}.$$

3.2 Relationship to Other Cost-Sharing Mechanisms

We first define coalitions to be able to explain why POCS is a combination of proportional and incremental cost sharing.

Definition 7. *For all times k_1, k_2 and t with $k_1 \leq k_2 \leq t$ and all arrival orders π, a coalition (k_1, k_2) at time t is a group of passengers $\pi(k_1), \ldots, \pi(k_2)$ with*

$$\frac{cost^t_{\pi(k)}}{\alpha_{\pi(k)}} = \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \quad \text{for all times } k \text{ with } k_1 \leq k \leq k_2$$

$$\frac{cost^t_{\pi(k)}}{\alpha_{\pi(k)}} \neq \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \quad \text{for all times } k \text{ with } (k = k_1 - 1 \text{ or } k = k_2 + 1) \text{ and } 1 \leq k \leq t.$$

We now state a lemma that we prove as Lemma 3 in Chapter 4. It states that the shared costs per alpha value of all passengers in any coalition are always identical and equal to the coalition cost per alpha value of the coalition.

Lemma 1. *The shared cost per alpha value of any passenger in any coalition at any time equals the coalition cost per alpha value of the coalition, that is, for all times k_1, k, k_2 and t with $1 \leq k_1 \leq k \leq k_2 \leq t$ such that (k_1, k_2) is a coalition at time t and all arrival orders π,*

$$\frac{cost^t_{\pi(k)}}{\alpha_{\pi(k)}} = ccpa_{\pi(k_1,k_2)}.$$

Lemma 1 implies that POCS is a combination of proportional and incremental cost sharing, for the following reasons:

- The sum of the marginal costs of all passengers in any coalition ("the total cost of all passengers in the coalition") at time t is distributed among all passengers in the coalition proportionally to their alpha values since, for all times k_1, k, k_2 and t with $k_1 \leq k \leq k_2 \leq t$ such that (k_1, k_2) is a coalition at time t and all arrival orders π,

$$cost^t_{\pi(k)} \overset{Lem.1}{=} \alpha_{\pi(k)} ccpa_{\pi(k_1,k_2)} \overset{Def.5}{=} \alpha_{\pi(k)} \frac{\sum_{j=k_1}^{k_2} mc_{\pi(j)}}{\sum_{j=k_1}^{k_2} \alpha_{\pi(j)}} = \left(\sum_{j=k_1}^{k_2} mc_{\pi(j)} \right) \frac{\alpha_{\pi(k)}}{\sum_{j=k_1}^{k_2} \alpha_{\pi(j)}},$$

  which is similar to proportional cost sharing, where the total cost (of all passengers) is distributed among all passengers proportionally to their alpha values.

- The sum of the shared costs of all passengers in any coalition ("the shared cost of the coalition") at time t equals the sum of the marginal costs of all passengers in the coalition ("the marginal cost of the coalition") at the same time since, for all times k_1, k_2 and t with $k_1 \leq k_2 \leq t$ such that (k_1, k_2) is a coalition at time t and all arrival orders π,

$$\sum_{j=k_1}^{k_2} cost^t_{\pi(j)} \overset{Lem.1}{=} ccpa_{\pi(k_1,k_2)} \sum_{j=k_1}^{k_2} \alpha_{\pi(j)} \overset{Def.5}{=} \frac{\sum_{j=k_1}^{k_2} mc_{\pi(j)}}{\sum_{j=k_1}^{k_2} \alpha_{\pi(j)}} \sum_{j=k_1}^{k_2} \alpha_{\pi(j)} = \sum_{j=k_1}^{k_2} mc_{\pi(j)},$$

  which is similar to incremental cost sharing. where the shared cost of a passenger is its marginal cost. It also implies the budget balance property since summing over all passengers in all coalitions is identical to summing over all passengers and the sum of the marginal costs of all passengers equals the total cost.

3.3 Illustration

Table 3.1 shows the coalition costs per alpha value for the DRT example in Section 2.2. The coalition costs per alpha value are used to calculate the shared costs, shown in Table 3.2. The shared costs, in turn, are used to calculate the shared costs per alpha

		$k_2 = 1$ $\pi(k_2) = P_1$	$k_2 = 2$ $\pi(k_2) = P_2$	$k_2 = 3$ $\pi(k_2) = P_3$	$k_2 = 4$ $\pi(k_2) = P_4$
$k_1 = 1$	$\pi(k_1) = P_1$	10	15	7 1/2	8
$k_1 = 2$	$\pi(k_1) = P_2$		20	6 2/3	7 1/2
$k_1 = 3$	$\pi(k_1) = P_3$			0	3 2/3
$k_1 = 4$	$\pi(k_1) = P_4$				10

Table 3.1: Coalition Costs per Alpha Value under POCS: $ccpa_{\pi(k_1,k_2)}$

	$k = 1$ $\pi(k) = P_1$	$k = 2$ $\pi(k) = P_2$	$k = 3$ $\pi(k) = P_3$	$k = 4$ $\pi(k) = P_4$
$t = 1$	20			
$t = 2$	20	40		
$t = 3$	15	15	30	
$t = 4$	15	15	30	20

Table 3.2: Shared Costs under POCS: $cost^t_{\pi(k)}$

value, shown in Table 3.3, by dividing the shared costs by the alpha values, shown in Table 2.1. For example, at time 4, Passengers P_1, P_2 and P_3 form a coalition (since their shared costs per alpha value are equal), and Passenger P_4 forms a coalition by itself. The sum of the marginal costs of the three passengers in the first coalition ("the total cost of all passengers in the coalition") is 60 and is distributed among all passengers in the coalition proportionally to their alpha values, namely 2, 2 and 4, respectively. Consequently, the shared cost of Passenger P_3 at time 4 and thus its fare is $cost^4_{\pi(3)} = 30$. Table 3.3 shows that the shared costs per alpha value in each row are monotonically non-decreasing from left to right, corresponding to the online fairness property. Table 3.2 shows that the shared costs in each column are monotonically non-increasing from top to bottom (and consequently Table 3.3 shows that the shared costs per alpha value have the same property), corresponding to the immediate response property. Table 3.2 also shows that the sum of the shared costs in each row equals the total cost at the corresponding time, corresponding to the budget balance property.

3.4 Ex-Post Incentive Compatibility

We use the DRT example in Figure 3.1 to illustrate that POCS does not satisfy the ex-post incentive compatibility property if the second condition (namely that none of the other passengers change whether they accept or decline their fare quotes) is removed.

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
	$\pi(k) = P_1$	$\pi(k) = P_2$	$\pi(k) = P_3$	$\pi(k) = P_4$
$t = 1$	10			
$t = 2$	10	20		
$t = 3$	7 1/2	7 1/2	7 1/2	
$t = 4$	7 1/2	7 1/2	7 1/2	10

Table 3.3: Shared Costs per Alpha Value under POCS: $cost^t_{\pi(k)}/\alpha_{\pi(k)}$

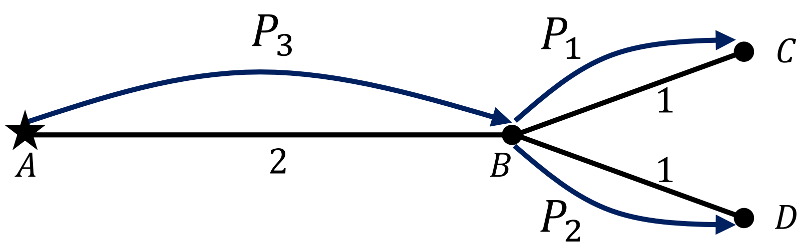
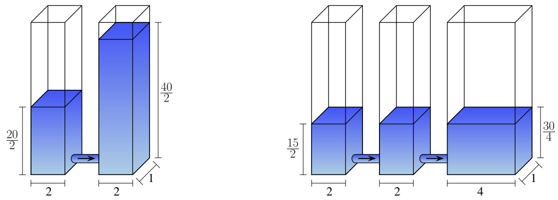


Figure 3.1: DRT Example 2

There is one shuttle that can transport up to four passengers and starts at the star. The shuttle incurs an operating cost of 10 for each unit of distance traveled and does not need to return to its initial location. There are three passengers. Passengers P_1 and P_3 accept all fare quotes, while Passenger P_2 accepts all fare quotes up to 27. Assume that the passengers arrive in order P_1, P_2 and P_3. First, Passenger P_1 arrives, receives a fare quote of 30 and accepts. Second, Passenger P_2 arrives, receives a fare quote of 25 and accepts. Third, Passenger P_3 arrives, receives a fare quote of 25 and accepts. In the end, Passengers P_1, P_2 and P_3 are serviced with fares of 12 1/2, 12 1/2 and 25, respectively. Now assume that Passenger P_1 delays its arrival, and the passengers arrive in order P_2, P_3 and P_1. First, Passenger P_2 arrives, receives a fare quote of 30 and drops out since the fare quote exceeds its limit of 27. Second, Passenger P_3 arrives, receives a fare quote of 20 and accepts. Third, Passenger P_1 arrives, receives a fare quote of 10 and accepts. In the end, Passengers P_1 and P_3 are serviced with fares of 10 and 20, respectively. Thus, Passenger P_1 managed to decrease both its fare quote and fare by delaying its arrival since this caused Passenger P_2 to drop out.



(a) $t = 2$ (b) $t = 3$

Figure 3.2: Water Analogy

3.5 Water Analogy

The following analogy using water in containers provided the inspiration for POCS. Every passenger is represented by a water container. The volume of water in the container corresponds to the shared cost of the passenger, the cross-sectional area of the container corresponds to the alpha value of the passenger, and the level of water in the container thus corresponds to the shared cost per alpha value of the passenger. When a new passenger arrives, it is allocated an empty container to the right of the existing containers. A volume of water that corresponds to the marginal cost of the passenger is poured into this new container, the new container is connected to the container of the previously arriving passenger with a one-way valve that allows water to flow only into the new container, and the water is allowed to settle and reach equilibrium. Afterwards, the volume of water in all containers corresponds to the total cost, and the volume of water in each container corresponds to the shared cost of the corresponding passenger. We explain the water analogy in following for the DRT example in Section 2.2. Figure 3.2(a) (left) shows the water levels in equilibrium after Passenger P_2 arrived and 40 units of water were poured into its container, and Figure 3.2(a) (right) shows the water levels in equilibrium after Passenger P_3 arrived and 0 units of water were poured into its container. In general, in equilibrium, the water levels are monotonically nondecreasing from left to right since water can flow only from left to right, corresponding to the on-line fairness property. The water level and thus also the volume of water in each container are monotonically non-increasing over time since water can flow only from left to right, corresponding to the immediate response property. With the exception of pouring water into new containers, water does not get added or removed, corresponding to the budget balance property.

4

Analysis of Properties

In this chapter, we prove that POCS satisfies all properties listed in Section 2.4, making use of the following corollary to Definition 6.

Corollary 1. *For all times k and t with $1 \leq k \leq t$ and all arrival orders π,*

$$\frac{cost^t_{\pi(k)}}{\alpha_{\pi(k)}} = \min_{k \leq j \leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}}.$$

Proof. Consider any times k and t with $1 \leq k \leq t$ and any arrival order π. Then,

$$\begin{aligned}
\frac{cost^t_{\pi(k)}}{\alpha_{\pi(k)}} &\overset{Def.6}{=} \min_{k \leq j \leq t} \max_{1 \leq i \leq j} ccpa_{\pi(i,j)} \\
&= \min_{k \leq j \leq t} \min_{j \leq j' \leq j} \max_{1 \leq i \leq j'} ccpa_{\pi(i,j')} \\
&\overset{Def.6}{=} \min_{k \leq j \leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}},
\end{aligned}$$

which proves the corollary. □

4.1 Online Fairness

In this section, we prove that POCS satisfies the online fairness property, namely that the shared costs per alpha value of passengers are never higher than those of passengers who arrive after them.

Theorem 1. *POCS satisfies the online fairness property, that is, for all times k_1, k_2 and t with $1 \le k_1 \le k_2 \le t$ and all arrival orders π,*

$$\frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \le \frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}}.$$

Proof. Consider any times k_1, k_2 and t with $1 \le k_1 \le k_2 \le t$ and any arrival order π. Then,

$$\frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \overset{Cor.1}{=} \min_{k_1 \le j \le t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \overset{k_1 \le k_2}{\le} \min_{k_2 \le j \le t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \overset{Cor.1}{=} \frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}},$$

which proves the theorem. □

4.2 Immediate Response

In this section, we prove that POCS satisfies the immediate response property, namely that passengers are provided at their arrival with upper bounds on their shared costs at any future time because they are provided with their shared costs at their arrival and their shared costs are monotonically non-increasing over time.

Theorem 2. *POCS satisfies the immediate response property, that is, for all times k, t_1 and t_2 with $1 \le k \le t_1 \le t_2$ and all arrival orders π,*

$$cost^{t_1}_{\pi(k)} \ge cost^{t_2}_{\pi(k)}.$$

Proof. Consider any times k, t_1 and t_2 with $1 \le k \le t_1 \le t_2$ and any arrival order π. Then,

$$cost^{t_1}_{\pi(k)} \overset{Cor.1}{=} \alpha_{\pi(k)} \min_{k \le j \le t_1} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \overset{t_1 \le t_2}{\ge} \alpha_{\pi(k)} \min_{k \le j \le t_2} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \overset{Cor.1}{=} cost^{t_2}_{\pi(k)},$$

which proves the theorem. □

4.3 Budget Balance

In this section, we prove that POCS satisfies the budget balance property, namely that the total cost equals the sum of the shared costs of all passengers. We first prove some properties of coalitions and then that the sum of the marginal costs of all passengers in any coalition ("the total cost of all passengers in the coalition") equals the sum of the shared costs of all passengers in it.

Lemma 2. *The shared cost of the last passenger in any coalition at any time equals its shared cost immediately after its arrival, that is, for all times k_1, k_2 and t with $1 \leq k_1 \leq k_2 \leq t$ such that (k_1, k_2) is a coalition at time t and all arrival orders π,*

$$cost^t_{\pi(k_2)} = cost^{k_2}_{\pi(k_2)}.$$

Proof. Consider any times k_1, k_2 and t with $1 \leq k_1 \leq k_2 \leq t$ such that (k_1, k_2) is a coalition at time t and any arrival order π. At time $t = k_2$, the lemma trivially holds. At time $t > k_2$,

$$\frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \overset{Th.1}{<} \frac{cost^t_{\pi(k_2+1)}}{\alpha_{\pi(k_2+1)}} \tag{4.1}$$

since passengers $\pi(k_2)$ and $\pi(k_2+1)$ are in different coalitions at time t and thus do not have the same shared cost per alpha value at time t according to Definition 7. Thus,[1]

$$\min_{k_2 \leq j \leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \overset{Cor.1}{=} \frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \overset{Eq.4.1}{<} \frac{cost^t_{\pi(k_2+1)}}{\alpha_{\pi(k_2+1)}} \overset{Cor.1}{=} \min_{k_2+1 \leq j \leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \tag{4.2}$$

$$cost^t_{\pi(k_2)} \overset{Cor.1}{=} \alpha_{\pi(k_2)} \min_{k_2 \leq j \leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \overset{Eq.4.2*}{=} \alpha_{\pi(k_2)} \frac{cost^{k_2}_{\pi(k_2)}}{\alpha_{\pi(k_2)}} = cost^{k_2}_{\pi(k_2)},$$

which proves the lemma. □

Lemma 3. *The shared cost per alpha value of any passenger in any coalition at any time equals the coalition cost per alpha value of the coalition, that is, for all times k_1, k, k_2 and t with $1 \leq k_1 \leq k \leq k_2 \leq t$ such that (k_1, k_2) is a coalition at time t and all arrival orders π,*

$$\frac{cost^t_{\pi(k)}}{\alpha_{\pi(k)}} = ccpa_{\pi(k_1,k_2)}.$$

Proof. Consider any times k_1, k, k_2 and t with $1 \leq k_1 \leq k \leq k_2 \leq t$ such that (k_1, k_2) is a coalition at time t and any arrival order π. We prove the lemma for $k = k_2$. It

[1] At the position marked with an asterisk, we use that $\min_{k_2 \leq j \leq t} x_j < \min_{k_2+1 \leq j \leq t} x_j$ implies $\min_{k_2 \leq j \leq t} x_j = x_{k_2}$.

then also holds for all k with $k_1 \leq k \leq k_2$ since all passengers in the same coalition at time t have the same shared cost per alpha value at time t according to Definition 7. We prove the lemma for $k = k_2$ by contradiction by assuming that

$$\frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \neq ccpa_{\pi(k_1,k_2)}. \tag{4.3}$$

Then,

$$ccpa_{\pi(k_1,k_2)} \overset{Eq.4.3}{\neq} \frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \overset{Lem.2}{=} \frac{cost^{k_2}_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \overset{Def.6}{=} \max\nolimits_{1\leq j\leq k_2} ccpa_{\pi(j,k_2)} \tag{4.4}$$

$$ccpa_{\pi(k_1,k_2)} \overset{Eq.4.4, 1\leq k_1\leq k_2}{<} \max\nolimits_{1\leq j\leq k_2} ccpa_{\pi(j,k_2)}. \tag{4.5}$$

Thus, there exists a time k' with $1 \leq k' < k_2$ such that

$$ccpa_{\pi(k_1,k_2)} \overset{Eq.4.5}{<} \max_{1\leq j\leq k_2} ccpa_{\pi(j,k_2)} = ccpa_{\pi(k',k_2)}. \tag{4.6}$$

Assume without loss of generality that k' is the earliest such time. Then, for all times k'' with $1 \leq k'' < k'$,

$$ccpa_{\pi(k'',k_2)} < ccpa_{\pi(k',k_2)}. \tag{4.7}$$

We distinguish the following cases to prove that such a k' does not exist. The cases are exhaustive since $1 \leq k' < k_2$ and $1 \leq k_1 \leq k_2$:

- **Case** $1 \leq k' < k_1$**:**
  Let $A := \sum_{j=k'}^{k_1-1} mc_{\pi(j)}$, $B := \sum_{j=k_1}^{k_2} mc_{\pi(j)}$, $C := \sum_{j=k'}^{k_1-1} \alpha_{\pi(j)}$ and $D := \sum_{j=k_1}^{k_2} \alpha_{\pi(j)}$. Then,

$$\tfrac{B}{D} \overset{Def.5}{=} ccpa_{\pi(k_1,k_2)} \overset{Eqs.4.6}{<} ccpa_{\pi(k',k_2)} \overset{Def.5}{=} \tfrac{A+B}{C+D} \tag{4.8}$$

$$BC \overset{Eq.4.8, C>0, D>0}{<} AD \tag{4.9}$$

$$ccpa_{\pi(k',k_2)} \overset{Def.5}{=} \tfrac{A+B}{C+D} \overset{Eq.4.9}{<} \tfrac{A}{C} \overset{Def.5}{=} ccpa_{\pi(k',k_1-1)}. \tag{4.10}$$

  Separately,

$$\frac{cost^t_{\pi(k_1-1)}}{\alpha_{\pi(k_1-1)}} \overset{Th.1}{<} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \tag{4.11}$$

  since passengers $\pi(k_1 - 1)$ and $\pi(k_1)$ are in different coalitions at time t and thus do not have the same shared cost per alpha value at time t according to Definition 7. Thus,

$$\min_{k_1-1\leq j\leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \overset{Cor.1}{=} \frac{cost^t_{\pi(k_1-1)}}{\alpha_{\pi(k_1-1)}} \overset{Eq.4.11}{<} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \overset{Cor.1}{=} \min_{k_1\leq j\leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \tag{4.12}$$

and[1]

$$
\begin{aligned}
\frac{cost^{k_1-1}_{\pi(k_1-1)}}{\alpha_{\pi(k_1-1)}} &\overset{Eq.4.12*}{<} \min_{k_1\le j\le t} \frac{cost^{j}_{\pi(j)}}{\alpha_{\pi(j)}} \\
&\overset{k_1\le k_2}{\le} \min_{k_2\le j\le t} \frac{cost^{j}_{\pi(j)}}{\alpha_{\pi(j)}} \\
&\overset{Cor.1}{=} \frac{cost^{t}_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \\
&\overset{Lem.2}{=} \frac{cost^{k_2}_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \\
&\overset{Def.6}{=} \max_{1\le j\le k_2} ccpa_{\pi(j,k_2)} \\
&\overset{Eq.4.6}{=} ccpa_{\pi(k',k_2)} \\
&\overset{Eq.4.10}{<} ccpa_{\pi(k',k_1-1)} \\
&\overset{1\le k'\le k_1-1}{\le} \max_{1\le j\le k_1-1} ccpa_{\pi(j,k_1-1)} \\
&\overset{Def.6}{=} \frac{cost^{k_1-1}_{\pi(k_1-1)}}{\alpha_{\pi(k_1-1)}},
\end{aligned}
$$

which is a contradiction since a value cannot be lower than itself.

- **Case** $k' = k_1$**:**
  It holds that

$$ccpa_{\pi(k',k_2)} \overset{k'=k_1}{=} ccpa_{\pi(k_1,k_2)} \overset{Eq.4.5}{<} \max_{1\le j\le k_2} ccpa_{\pi(j,k_2)} \overset{Eq.4.6}{=} ccpa_{\pi(k',k_2)},$$

which is a contradiction since a value cannot be lower than itself.

- **Case** $k_1 < k' < k_2$**:**
  Let $A := \sum_{j=k''}^{k'-1} mc_{\pi(j)}$, $B := \sum_{j=k'}^{k_2} mc_{\pi(j)}$, $C := \sum_{j=k''}^{k'-1} \alpha_{\pi(j)}$ and $D := \sum_{j=k'}^{k_2} \alpha_{\pi(j)}$. Then, for all times k'' with $1 \le k'' < k'$,

$$\frac{A+B}{C+D} \overset{Def.5}{=} ccpa_{\pi(k'',k_2)} \overset{Eq.4.7}{<} ccpa_{\pi(k',k_2)} \overset{Def.5}{=} \frac{B}{D} \tag{4.13}$$

$$AD \overset{Eq.4.13,C>0,D>0}{<} BC \tag{4.14}$$

$$ccpa_{\pi(k'',k'-1)} \overset{Def.5}{=} \frac{A}{C} \overset{Eq.4.14}{<} \frac{B}{D} \overset{Def.5}{=} ccpa_{\pi(k',k_2)}. \tag{4.15}$$

[1] At the position marked with an asterisk, we use that $\min_{k_1-1\le j\le t} x_j < \min_{k_1\le j\le t} x_j$ implies $x_{k_1-1} < \min_{k_1\le j\le t} x_j$.

Thus,[1]

$$\begin{aligned}
\frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} &\overset{Cor.1}{=} \min_{k_1 \le j \le t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \\
&\overset{k_1 \le k'-1 < t}{\le} \frac{cost^{k'-1}_{\pi(k'-1)}}{\alpha_{\pi(k'-1)}} \\
&\overset{Def.6}{=} \max_{1 \le j \le k'-1} ccpa_{\pi(j,k'-1)} \\
&\overset{Eq.4.15*}{<} ccpa_{\pi(k',k_2)} \\
&\overset{Eq.4.6}{=} \max_{1 \le j \le k_2} ccpa_\pi(j,k_2) \\
&\overset{Def.6}{=} \frac{cost^{k_2}_{\pi(k_2)}}{\alpha_{\pi(k_2)}} \\
&\overset{Lem.2}{=} \frac{cost^t_{\pi(k_2)}}{\alpha_{\pi(k_2)}},
\end{aligned}$$

which is a contradiction since passengers $\pi(k_1)$ and $\pi(k_2)$ are in the same coalition at time t and thus have the same shared cost per alpha value at time t according to Definition 7.

□

Theorem 3. *POCS satisfies the budget balance property, that is, for all times $1 \le t$ and all arrival orders π,*

$$\sum_{j=1}^{t} cost^t_{\pi(j)} = totalcost^t_\pi.$$

Proof. Consider any times k_1, k_2 and t with $1 \le k_1 \le k_2 \le t$ such that (k_1, k_2) is a coalition at time t and any arrival order π. Then,

$$\sum_{j=k_1}^{k_2} cost^t_{\pi(j)} \overset{Lem.3}{=} ccpa_{\pi(k_1,k_2)} \sum_{j=k_1}^{k_2} \alpha_{\pi(j)} \overset{Def.5}{=} \frac{\sum_{j=k_1}^{k_2} mc_{\pi(j)}}{\sum_{j=k_1}^{k_2} \alpha_{\pi(j)}} \sum_{j=k_1}^{k_2} \alpha_{\pi(j)} = \sum_{j=k_1}^{k_2} mc_{\pi(j)} \tag{4.16}$$

Summing over all passengers in all coalitions is identical to summing over all passengers. Thus,

$$\sum_{j=1}^{t} cost^t_{\pi(j)} \overset{Eq.4.16}{=} \sum_{j=1}^{t} mc_{\pi(j)} \overset{Def.3}{=} \sum_{j=1}^{t} \left(totalcost^t_\pi - totalcost^{t-1}_\pi\right) = totalcost^t_\pi,$$

[1] At the position marked with an asterisk, we use that, for all k'' with $1 \le k'' < k'$, $x_{k''} < y$ implies $\max_{1 \le k'' < k'} x_{k''} < y$.

which proves the theorem. $\square$

4.4 Ex-Post Incentive Compatibility

In this section, we prove that POCS satisfies the ex-post incentive compatibility (EPIC) property, namely that the best strategy for every passenger is to arrive truthfully provided that all other passengers arrive truthfully as well and do not change whether they accept or decline their fare quotes because it then cannot decrease its shared cost by delaying its arrival.

Theorem 4. *POCS satisfies the incentive compatibility (EPIC) property, that is, for all times k_1, k_2 and t with $1 \leq k_1 < k_2 \leq t$ and all arrival orders π and π' with*

$$\pi'(k) = \begin{cases} \pi(k+1) & \text{if } k_1 \leq k < k_2 \\ \pi(k_1) & \text{if } k = k_2 \\ \pi(k) & \textit{otherwise,} \end{cases} \tag{4.17}$$

$$cost^t_{\pi(k_1)} \leq cost^t_{\pi'(k_2)}.$$

Proof. Consider any times k_1, k_2 and t with $1 \leq k_1 < k_2 \leq t$ and any arrival orders π and π' with $\pi'(k)$ as given in Equation 4.17. π is the arrival order where every passenger arrives truthfully and the passengers arrive in order $\pi(1), \pi(2), \ldots, \pi(k_1 - 1), \pi(k_1), \pi(k_1+1), \ldots, \pi(k_2-1), \pi(k_2), \pi(k_2+1), \ldots, \pi(t)$, while π' is the arrival order where passenger $\pi(k_1)$ delays its arrival and the passengers arrive in order $\pi(1), \pi(2), \ldots, \pi(k_1 - 1), \pi(k_1+1), \ldots, \pi(k_2-1), \pi(k_2), \pi(k_1), \pi(k_2+1), \ldots, \pi(t)$. We prove the theorem by contradiction by assuming that

$$cost^t_{\pi'(k_2)} \quad < \quad cost^t_{\pi(k_1)}. \tag{4.18}$$

Then,

$$\frac{cost^t_{\pi'(k_2)}}{\alpha_{\pi'(k_2)}} \overset{Eq.4.18}{<} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}}, \tag{4.19}$$

since the alpha values do not depend on the arrival order according to Definition 1 and thus $\alpha_{\pi'(k_2)} = \alpha_{\pi(k_1)}$. Assume without loss of generality that t is the earliest such time. Thus, if $k_2 < t$,

$$\frac{cost^{t-1}_{\pi'(k_2)}}{\alpha_{\pi'(k_2)}} \geq \frac{cost^{t-1}_{\pi(k_1)}}{\alpha_{\pi(k_1)}}. \tag{4.20}$$

Separately,

$$\sum_{j=1}^{t} cost^t_{\pi'(j)} \overset{Th.3}{=} totalcost^t_{\pi'} = totalcost^t_{\pi} \overset{Th.3}{=} \sum_{j=1}^{t} cost^t_{\pi(j)} \tag{4.21}$$

due to the budget balance property since the total cost at time t does not depend on the arrival order according to Definition 2. Equations 4.20 and 4.21 together imply that there exists a passenger $\pi(m)$ with $1 \leq m \leq t$ and $m \neq k_1$ such that the shared cost of passenger $\pi(m)$ at time t is higher under arrival order π' than arrival order π because the shared cost of passenger $\pi(k_1)$ at time t is lower under arrival order π' than arrival order π (**Existence Property**). We distinguish the following cases to prove that such a passenger does not exist due to the online fairness and immediate response properties. The cases are exhaustive since $1 \leq m \leq t$, $m \neq k_1$ and $1 \leq k_1 < k_2 \leq t$:

- **Case** $1 \leq m < k_1$**:**
  In this case, $\pi'(m) \overset{Eq.4.17}{=} \pi(m)$. We distinguish the following subcases. They are exhaustive since
  $$\frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}} \overset{Th.1}{\leq} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}}.$$
  - **Sub-Case** $\frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}} < \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}}$**:**
    It holds that
    $$\begin{aligned} \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}} &\overset{Cor.1}{=} \min_{m \leq j \leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \\ &= \min\left(\min_{m \leq j \leq k_1-1} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}}, \min_{k_1 \leq j \leq t} \frac{cost^j_{\pi(j)}}{\alpha_{\pi(j)}} \right) \\ &\overset{Cor.1}{=} \min\left(\frac{cost^{k_1-1}_{\pi(m)}}{\alpha_{\pi(m)}}, \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \right) \end{aligned} \tag{4.22}$$
    Thus,[1]
    $$\frac{cost^{k-1}_{\pi(m)}}{\alpha_{\pi(m)}} \overset{Eq.4.22*,subcase-assumption}{=} \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}}. \tag{4.23}$$
    Put together,
    $$\frac{cost^t_{\pi'(m)}}{\alpha_{\pi'(m)}} \overset{Th.2}{\leq} \frac{cost^{k_1-1}_{\pi'(m)}}{\alpha_{\pi'(m)}} \overset{\forall_{1 \leq k < k_1} \pi(k)=\pi'(k)}{=} \frac{cost^{k_1-1}_{\pi(m)}}{\alpha_{\pi(m)}} \overset{Eq.4.23}{=} \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}},$$

[1] At the position marked with an asterisk, we use that $x < z$ and $x = \min(y, z)$ implies $x = y$.

which contradicts the Existence Property, namely that there exists a passenger $\pi(m) = \pi'(m)$ with

$$\frac{cost^t_{\pi'(m)}}{\alpha_{\pi'(m)}} > \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}}.$$

– **Sub-Case** $\frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}} = \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}}$**:**
It holds that

$$\frac{cost^t_{\pi'(m)}}{\alpha_{\pi'(m)}} \overset{Th.1}{\leq} \frac{cost^t_{\pi'(k_2)}}{\alpha_{\pi'(k_2)}} \overset{Eq.4.19}{<} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \overset{subcase-assumption}{=} \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}},$$

which contradicts the Existence Property, namely that there exists a passenger $\pi(m) = \pi'(m)$ with

$$\frac{cost^t_{\pi'(m)}}{\alpha_{\pi'(m)}} > \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}}.$$

- **Case** $k_1 < m \leq k_2$**:**
In this case, $\pi'(m-1) \overset{Eq.4.17}{=} \pi(m)$. It holds that

$$\frac{cost^t_{\pi'(m-1)}}{\alpha_{\pi'(m-1)}} \overset{Th.1}{\leq} \frac{cost^t_{\pi'(k_2)}}{\alpha_{\pi'(k_2)}} \overset{Eq.4.19}{<} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \overset{Th.1}{\leq} \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}},$$

which contradicts the Existence Property, namely that there exists a passenger $\pi(m) = \pi'(m-1)$ with

$$\frac{cost^t_{\pi'(m-1)}}{\alpha_{\pi'(m-1)}} > \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}}.$$

- **Case** $k_2 < m \leq t$**:**
In this case, $\pi'(m) \overset{Eq.4.17}{=} \pi(m)$. It holds that

$$\begin{aligned} \min_{k_2 \leq j \leq t} \frac{cost^j_{\pi'(j)}}{\alpha_{\pi'(j)}} &\overset{Cor.1}{=} \frac{cost^t_{\pi'(k_2)}}{\alpha_{\pi'(k_2)}} \\ &\overset{Eq.4.19}{<} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \\ &\overset{Th.2}{\leq} \frac{cost^{t-1}_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \\ &\overset{Eq.4.20}{\leq} \frac{cost^{t-1}_{\pi'(k_2)}}{\alpha_{\pi'(k_2)}} \\ &\overset{Cor.1}{=} \min_{k_2 \leq j \leq t-1} \frac{cost^j_{\pi'(j)}}{\alpha_{\pi'(j)}} \end{aligned} \tag{4.24}$$

and thus[1]

$$\frac{cost^t_{\pi'(t)}}{\alpha_{\pi'(t)}} \overset{Eq.4.24*}{<} \min_{k_2 \le j \le t-1} \frac{cost^j_{\pi'(j)}}{\alpha_{\pi'(j)}}. \tag{4.25}$$

Therefore,[2]

$$\begin{aligned}
\frac{cost^t_{\pi'(m)}}{\alpha_{\pi'(m)}} &\overset{Th.1}{\le} \frac{cost^t_{\pi'(t)}}{\alpha_{\pi'(t)}} \\
&\overset{Eq.4.25*}{=} \min\left(\min_{k_2 \le j \le t-1} \frac{cost^j_{\pi'(j)}}{\alpha_{\pi'(j)}}, \frac{cost^t_{\pi'(t)}}{\alpha_{\pi'(t)}}\right) \\
&= \min_{k_2 \le j \le t} \frac{cost^j_{\pi'(j)}}{\alpha_{\pi'(j)}} \\
&\overset{Cor.1}{=} \frac{cost^t_{\pi'(k_2)}}{\alpha_{\pi'(k_2)}} \\
&\overset{Eq.4.19}{<} \frac{cost^t_{\pi(k_1)}}{\alpha_{\pi(k_1)}} \\
&\overset{Th.1}{\le} \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}},
\end{aligned}$$

which contradicts the Existence Property, namely that there exists a passenger $\pi(m) = \pi'(m)$ with

$$\frac{cost^t_{\pi'(m)}}{\alpha_{\pi'(m)}} > \frac{cost^t_{\pi(m)}}{\alpha_{\pi(m)}}.$$

□

[1] At the position marked with an asterisk, we use that $\min_{k_2 \le j \le t} x_j < \min_{k_2 \le j \le t-1} x_j$ implies $x_t < \min_{k_2 \le j \le t-1} x_j$.

[2] At the position marked with an asterisk, we use that $x < y$ implies $x = \min(y, x)$.

5

Conclusions

In this report, we determined properties of cost-sharing mechanisms that we believe make demand responsive transit systems attractive to both providers and passengers, namely online fairness, immediate response, budget balance and ex-post incentive compatibility. The online fairness property is meant to ensure that passengers consider the fares to be fair although this still needs to be tested empirically. We then proposed a novel cost-sharing mechanism, called Proportional Online Cost Sharing (POCS), and used an analogy involving water in containers to motivate it. POCS provides passengers with upper bounds on their fares immediately after their arrival, allowing the passengers to accept or decline. Thus, passengers have no uncertainty about whether they can be serviced or how high their fares will be at most, while the DRT systems reduce their uncertainty about passengers dropping out and can thus prepare better. Yet, they still retain some flexibility to optimize the routes and schedules after future arrivals. The sum of the fares of all passengers always equals the operating cost. Thus, no profit is made and no subsidies are required. POCS provides incentives for passengers to arrive truthfully since the fares of passengers per mile of requested travel are never higher than those of passengers who arrive after them. Thus, the DRT systems have more time to prepare and might also be able to offer subsequent passengers lower fares due to synergies with the early ride requests, which might allow them to service more passengers. Passengers also have incentives to arrive truthfully since the likelihood of transportation capacity being available tends to decrease over time, which alleviates the issue that the best strategy of every passenger is to arrive truthfully only under assumptions that are only approximately satisfied by POCS.

Overall, POCS is a very first step toward addressing some of the problems raised by missing knowledge of future arrivals, which differentiates our research from previous research (1, 8, 14, 23). However, lots of issues remain to be addressed by more advanced online cost-sharing mechanisms, including integrating more complex models of passengers, shuttles and transit environments. Our current simplifying assumptions include, for example, that the availability of shuttles does not change unexpectedly, that all passengers arrive before the shuttles start to service passengers, that fares depend only on the ride requests and no other considerations (for example, that DRT systems do not face competition), that all passengers evaluate their trips uniformly according to the criteria quantified by the alpha values (for example, that all passengers consider travel time to be equally important), that DRT systems provide fare quotes to passengers without predicting future arrivals (for example, that DRT systems do not reject hard-to-accommodate passengers even though they increase the shared costs of subsequent passengers and might make subsequent passengers drop out), that passengers try to decrease their fares only by delaying their arrival (rather than, for example, by colluding with other passengers or entering fake ride requests under false names) and that passengers honor their commitments (for example, that passengers do not change ride requests, cancel them, show up late or do not show up at all). Finally, it is future work to integrate POCS into auctions that determine the routes and schedules of shuttles and ensure that the resulting operating costs satisfy our assumptions (or, alternatively, relax the assumptions).

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