FINAL REPORT

to

THE FLORIDA DEPARTMENT OF TRANSPORTATION
RESEARCH OFFICE

on Project

"Nonlinear Road Pricing"

FDOT Contract BDK77 977-12 (UF Project 00093713)

August, 2012

Transportation Research Center
The University of Florida

Disclaimer

The opinions, findings, and conclusions expressed in this publication are those of the authors and not necessarily those of the State of Florida Department of Transportation.

Metric Conversion Chart

U.S. units to metric (SI) units

Length

Symbol	When you know	Multiply by	To find	Symbol
in	inches	25.4	millimeters	mm
ft	feet	0.305	meters	m
yd	yards	0.914	meters	m
mi	miles	1.61	kilometers	km

Metric (SI) units to U.S. units

Length

Symbol	When you know	Multiply by	To find	Symbol
mm	millimeters	0.039	inches	in
m	meters	3.28	feet	ft
m	meters	1.09	yards	yd
km	kilometers	0.621	miles	mi

Technical Report Documentation Page

<table>
<tr><td colspan="2">1. Report No.</td><td colspan="2">2. Government Accession No.</td><td colspan="2">3. Recipient's Catalog No.</td></tr>
<tr><td colspan="4" rowspan="2">4. Title and Subtitle

Nonlinear Road Pricing</td><td colspan="2">5. Report Date

August, 2012</td></tr>
<tr><td colspan="2">6. Performing Organization Code
UF-TRC</td></tr>
<tr><td colspan="4">7. Author(s)
Siriphong Lawphongpanich and Yafeng Yin</td><td colspan="2">8. Performing Organization Report No.
UF-TRC-79803-2011</td></tr>
<tr><td colspan="4" rowspan="2">9. Performing Organization Name and Address
Transportation Research Center
University of Florida
512 Weil Hall, PO Box 116580
Gainesville, FL 32611-6580</td><td colspan="2">10. Work Unit No. (TRAIS)</td></tr>
<tr><td colspan="2">11. Contract or Grant No.
FDOT Contract BDK77 977-12</td></tr>
<tr><td colspan="4" rowspan="2">12. Sponsoring Agency Name and Address
Florida Department of Transportation
605 Suwannee Street, MS 30
Tallahassee, FL 32399</td><td colspan="2">13. Type of Report and Period Covered

Draft Final, 3/25/2011–10/31/2012</td></tr>
<tr><td colspan="2">14. Sponsoring Agency Code</td></tr>
<tr><td colspan="6">15. Supplementary Notes</td></tr>
<tr><td colspan="6">16. Abstract
Nonlinear pricing refers to a case in which the price or tariff is not strictly proportional to the quantity purchased. While economists have studied nonlinear pricing for quite some time, its application to road pricing is relatively unexplored in the transportation literature. The number of articles on nonlinear road pricing is few, and many address only its impacts via empirical evidence. There has been little attempt to determine an optimal nonlinear pricing scheme, e.g., that maximizes the social welfare, especially for large road networks.
The objective of this research is to develop methodologies for determining optimal nonlinear road pricing schemes for realistic road networks and explore its impacts, e.g., on congestion, equity, and other factors.
In this study, we establish new results concerning nonlinear road pricing. In particular, the conditions under which link-based equilibrium conditions exist are of particularly importance in theory. New and efficient algorithms for determining optimal pricing structures are developed. These algorithms are useful to various transportation agencies and private companies in developing and analyzing nonlinear pricing schemes for the roads under their jurisdiction.</td></tr>
<tr><td colspan="3">17. Key Word
Congestion pricing, Nonlinear Pricing, VMT fee</td><td colspan="3">18. Distribution Statement
No restrictions</td></tr>
<tr><td>19. Security Classification of this report: Unclassified.</td><td colspan="2">20. Security Classification of this page: Unclassified.</td><td colspan="2">21. No. of Pages: 82</td><td>22. Price
N/A</td></tr>
</table>

Form DOT F 1700.7 (8-72) Reproduction of completed page authorized

Acknowledgments

The authors would like to express their sincere appreciation to Mr. Martin Markovich of the Florida Department of Transportation for the guidance and support that he provided on this project.

In addition, the authors would like to acknowledge the assistance they received from Mr. Darryll Dockstader, Ms. Sandra Bell, and Ms. Patti Brannon from the Research Center, Florida Department of Transportation, and Dr. Lily Elefteriadou, Ms. Ines Aviles-Spadoni and Ms. Dona Moss from the Center for Multimodal Solutions for Congestion Mitigation, University of Florida, during all phases of this project.

Executive Summary

This report considers a form of road pricing that charges a usage fee based on the distance (measured in miles) traveled inside tolling areas. Often, a tolling project or program in a city or region involves charging for road usage inside one contiguous area, typically the central business district. As illustrated in the appendix, it is possible to allow tolling, particularly with the same fee structure, to occur in multiple areas that are not necessarily connected, adjacent and/or contiguous. In one extreme, a tolling area can consist of every road inside a county, state or the entire country. Doing so is similar to the mileage fee that is a subject of the current discussion concerning a possible replacement for the gas tax. At the opposite extreme, a tolling area can be a single highway such as the Florida's Turnpike (also designated as SR 91). Other than these two extremes, the tolling areas, e.g., can consist of the (not necessarily connected) eight roadways comprising the Florida's Turnpike system or several areas inside a large metropolis.

To allow for more flexibility, the fee structures considered in this report are nonlinear, i.e., the fee charged depends on the distance traveled in a nonlinear fashion. Simply put, nonlinear pricing, or pricing with a nonlinear fee structure, refers to a case in which the price, tariff or fee charged may not be strictly proportional to the quantity purchased. Economists have been studying such form of pricing as early as 1894. Today, nonlinear pricing schemes are prevalent in many industries. For example, railroad tariffs generally depend on the weight, volume, and distance of each shipment. However, those using full cars and/or over long distances often receive discounts. The price per kilowatt-hour of electricity is different for different types of users. Heavy users during peak hours generally pay higher rates. Airlines routinely offer discount tickets for advance purchase, with noncancellation restriction, and in competitive markets. In each of these examples, the average price paid per unit varies depending on characteristics of the purchase, such as its size, time of usage, and restrictions and not necessarily proportional to the shipment distance, the amount used or the travel distance.

A mileage fee (often referred to as a vehicle miles traveled or VMT fee) is linear in structure because the amount charged is proportional to the distance traveled. In theory, a linear fee structure is a special case of nonlinear fee structures. Therefore, nonlinear road pricing includes mileage or VMT fees as well. However, the term nonlinear pricing in this report generally refers to pricing that is not linear or pricing other than mileage fees.

Road pricing in practice is often nonlinear. For example, toll prices in the central business district of London and Stockholm are not proportional to the distance traveled in the tolling area. Motorists in London have to pay a fixed rate of £8 per day to enter and use roads in the central business district. However, London offers monthly and annual passes to its frequent users at an approximately 15% discount. In Stockholm, motorists have to pay a toll each time they enter the tolling area—the central business district. The amount of tolls paid on a given day is limited to SEK 60. When the maximum amount is reached, motorists can freely enter the area. Thus, the charge in Stockholm is not necessarily proportional to the number of entries or distance traveled. Despite its relatively pervasive use, the literature on nonlinear road pricing is limited. Several authors have observed that the transportation literature has largely overlooked nonlinear road pricing. Our own literature survey netted less than ten journal articles on the subject.

This report examines nonlinear pricing at two levels of details: micro and macro. The latter is more suitable for policy decisions. The former is better suited for a more detailed impact analysis at a local level such as city or areas therein. The results for the micro-level analysis in this report involve new theoretical results and methodologies not previously published in the literature. Below summarizes the results from our examination of nonlinear pricing at these two levels.

To assess the impacts of nonlinear pricing at a macro level, this study employed a log-log linear regression model to estimate the annual travel demand (or annual vehicle miles traveled) per household (HH) from factors such as fuel cost, annual income, numbers of employed HH members, children, and vehicles, and household locations (e.g., rural or urban). The 2009 data for HH in Florida from the National Household Travel Survey were used to estimate the parameters for the regression model. After removing incomplete and/or illogical entries, the database for our parameter estimation consists of 13,086 HH with the following characteristics:

- The average fuel efficiencies of vehicles owned by rural and urban HH are approximately the same, approximately 21 miles per gallon (MPG). The similar is also true for HH at different levels of income.

- The annual VMT for rural HH (25,552 miles per HH) is higher than that of urban HH (19,905 miles per HH).

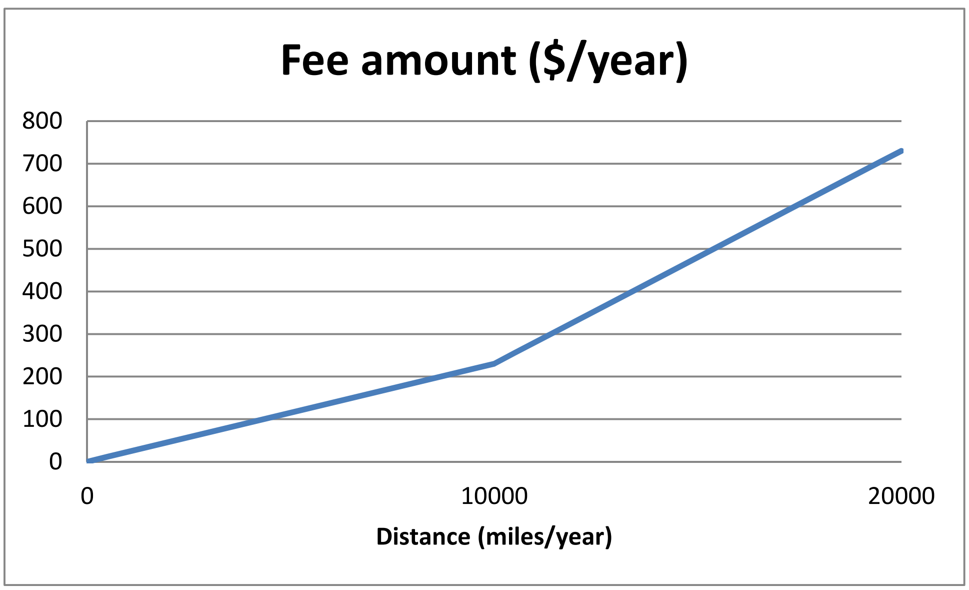
- Among the vehicles owned by rural HH, approximately 43% and 45% are cars and SUV[1]/trucks, respectively. For urban HH, the percentages are 55% and 39% instead.

To establish a basis for our macro-level analysis, we considered replacing the current state and county gas tax of 34.5 cents per gallon with a mileage fee. Using the fuel efficiency of 21 MPG, charging a fee of 1.64 cents per mile should generate the same amount of revenue as the 34.5¢ tax. In other words, the 1.64¢ mileage fee should be a *revenue-neutral* fee. On the other hand, our regression model indicates that 1.61¢ mileage fee is revenue-neutral instead. This is because the model accounts for changes in the annual VMT due to replacing gas tax with the mileage fee.

[1] Sport Utility Vehicles

In an effort to reduce the HH's annual VMT (and indirectly the congestion level), we also investigated the impacts of increasing the mileage fee to 2.8¢ and 4.1¢ per mile, respectively. Doing so reduces the number of miles traveled by our 13,086 households by 10 and 20 percent and increases the toll revenue by $2.3M and $4.4M, respectively. As expected, higher fees reduce the consumer surplus (i.e., cost or time-savings to motorists) among all different levels of income and the two categories of locations. As suggested by the 0.06 Gini coefficient, the reductions in consumer surplus are approximately equal across the various counties in Florida. However, the same is not true across the different levels. As a percentage of the annual income, the reduction in consumer surplus associated with 2.8¢ fee is 0.84% among HH with the lowest income (less than $20K annually) while the reduction is only 0.22% for those with the highest income (between $80K and $200K annually). For the 4.1¢ mileage fee, the reductions are 1.66% and 0.47% for lowest and highest income group, respectively. This makes mileage fees higher than the revenue-neutral level *regressive*. In other words, raising the flat mileage fee beyond the revenue-neutral level negatively affects the lower income groups more than those with higher incomes.

To illustrate the benefits on nonlinear pricing, consider a nonlinear fee structure depicted in the figure below. It displays a fee structure that charges a lower mileage fee (at 2.3¢ per mile in the figure) when the annual VMT is less than a threshold (10K annually). The fee of each mile above 10K is higher (at 5.0¢ per mile in the figure). In this report, this type of fees is referred to as 2-VMT fees.



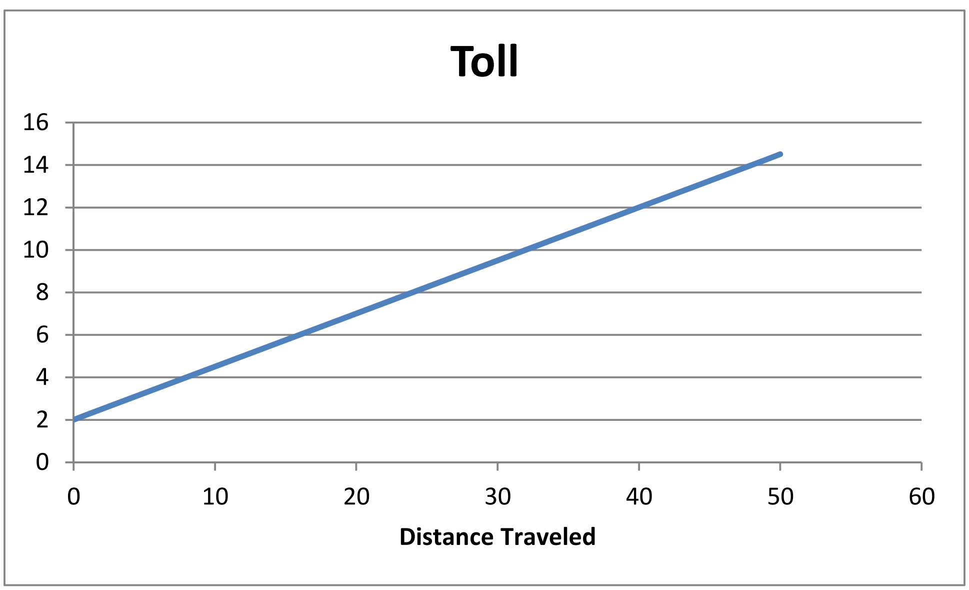
A pricing function

In our study, we investigated three 2-VMT schemes, all of which have the lower fees between 2.30¢ and 2.35¢ per mile, the thresholds between 10K to 15K annually, and the higher fee between 3.50¢ and 4.10¢ per mile. When compared to the case with one mileage fee, all three 2-VMT schemes are less regressive. While the percent reduction in consumer surplus for the highest income group remains approximately 0.28$, all three 2-VMT schemes reduce the consumer surplus for the lowest income group by no more than 0.67% (instead of 0.84% and 1.66% reduction reported above for 2.8¢ and 4.1¢ mileage fee). This suggests that simple nonlinear pricing schemes such as 2-VMT fees are less regressive.

We found only six journal articles addressing nonlinear pricing at the micro level. Four articles allow the pricing function to be any nonlinear function and design special algorithms to examine their impacts on, e.g., social surplus, congestion and pollution level. The remaining two considers that the case when the pricing function consisting solely of an entry fee similar to the congestion charging scheme in London. As advocated by many economists, this report only considers simple pricing functions because it is easier for motorists to understand and appropriately react to simple functions. Moreover, simple pricing functions (as demonstrated in the appendix) also lead to simpler algorithms for determining impacts on motorists/consumers.

For micro-level analyses, the contributions of this project are technical in nature. Our literature survey discovered that methodologies suitable for nonlinear pricing in the literature are either complex or ineffective for realistic road networks. Unlike articles in the literature that allow general nonlinear functions for determining toll amounts, our study focused on simple nonlinear functions such as the following:

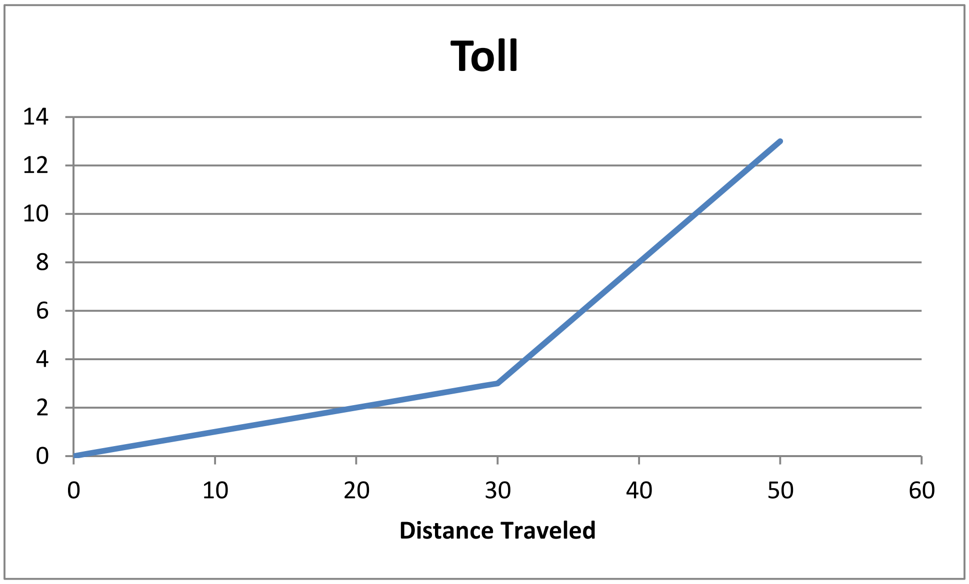
Two-part pricing: The figure below is the graph of the tolling function $T(\ell) = 2.0 + 0.25\ell$, where ℓ represents the distance traveled inside the tolling areas. In words, $T(\ell)$ charges motorists $2.0 to access the tolling areas and $0.25 per mile traveled (per day[2]) inside these areas.



Two-part pricing

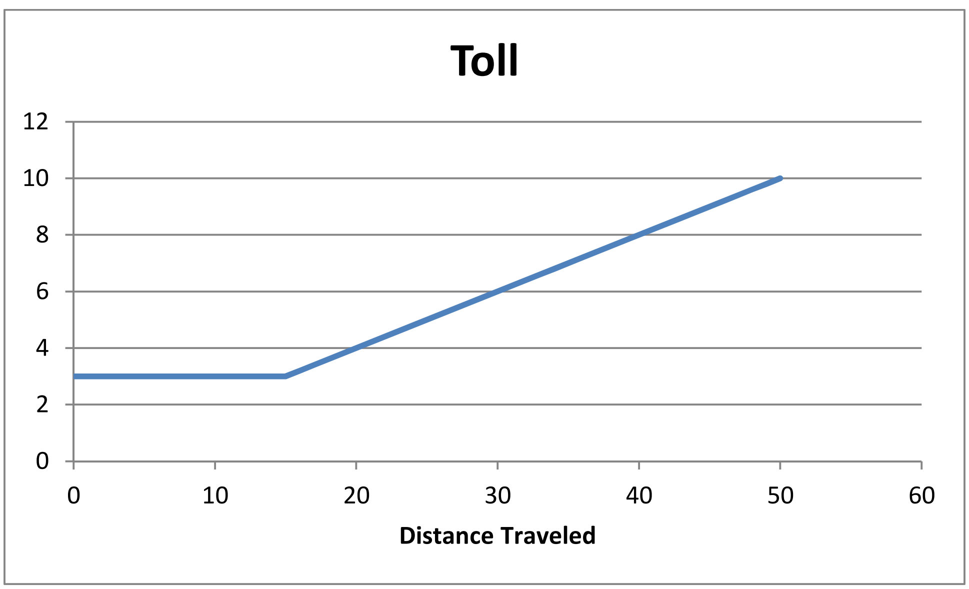
[2] It is also possible to replace "per day" with "per year" without affecting the results in the report.

Two-VMT pricing: The figure below is a graph of the tolling function $T(\ell) = \max\{0.1\ell, 0.5\ell - 12\}$. Under this function, the VMT fee is \$0.10 per mile if ℓ, the distance traveled (per day), is no more than 30 miles. When ℓ is larger than 30 miles, the fee is \$0.50 per mile traveled in excess of 30. The latter should discourage heavy road usage.



Pricing function for discouraging heavy road usage or 2-VMT pricing

Three-part pricing: The figure below displays the graph of the pricing function $T(\ell) = \max\{3.0, 0.2\ell\}$. Under this three-part pricing, motorists are charged \$3 access fee. Associated with this fee, motorists can travel up to 15 miles (per day) for free. The travel distance exceeding 15 miles is charged at a rate of \$0.20 per mile.



Three-part pricing

For the above three nonlinear pricing schemes, the report offers the following

- An algorithm/method for determining the effect of each pricing scheme on traffic flows. This algorithm can address realistic road networks and uses traditional techniques for computing traffic equilibria. The latter is advantageous because algorithms in existing software, commercial or otherwise, can be easily modified to include nonlinear pricing.

- An algorithm/method to search for an optimal nonlinear pricing scheme. This algorithm relies on the method in the above bullet and guarantees to find locally optimal solutions.

To illustrate, this report considers an actual road network from Hull, Canada, a test network well-known in the literature. This network contains 501 nodes (of which 23 are centroids), 789 links and 142 origin-destination pairs. The social surplus[3] for Hull without any pricing intervention is 6,935.13 units while the maximum obtainable social surplus is 17,902.45. For the optimal linear and nonlinear pricing, the results are listed below:

- Under a (locally) optimal linear pricing (i.e., $T(l) = 5.0\ell$), the social the social surplus is 7,503.06 units. When compared to the case without any tolling intervention, linear pricing increases the social surplus from 6,935.13 to 7,503.06 or by approximately 8%.

- Under a (locally) optimal two-part pricing scheme (i.e., $T(\ell) = 0.5 + 0.5\ell$), the social surplus is 7,770.17 units. When compared to the case without any tolling intervention, two-part pricing increases the social surplus from 6,935.13 to 7,770.17 or by approximately 12%.

- Under a (locally) optimal two-VMT pricing scheme (i.e., $T(\ell) = \max\{0.25, -1.0 + 3.1\ell\}$), the social surplus is 7,021.89 units. When compared to the case without any tolling intervention, two-VMT pricing increases the social surplus from 6,935.13 to 7,021.89 or by approximately 1.2%.

Among the three pricing schemes tested, the optimal two-part scheme achieved the best social benefit the Hull network. However, the above results may not be generalized to other cities. In general, the optimal social benefit depends on the topology of the road network, the tolling areas, travel demands, etc.

---

[3] In economics, the social surplus is the sum of consumer and producer surplus. In our setting, the consumer surplus is the amount of saving (in time or monetary term) realized by users or motorists. Similarly, the producer surplus is the benefit (in time or monetary term) gained by producers, i.e., government agencies.

Table of Contents

List of Figures

List of Tables

1 Introduction

Nonlinear pricing generally refers to a case in which the price or tariff is not strictly proportional to the quantity purchased. Economists have been studying such pricing since the discussion of its manifestations in Dupuit (1894) and the later categorization of the phenomenon in Pigou (1920). Today, nonlinear pricing is prevalent in many industries. For example, railroad tariffs generally depend on the weight, volume, and distance of each shipment. However, those using full cars and/or over long distances often receive discounts. The price per kilowatt-hour of electricity is different for different types of users. Heavy users during peak hours generally pay higher rates. Airlines routinely offer discount tickets for advance purchase, with noncancellation restriction, and in competitive markets. In each of these examples, the average price paid per unit varies depending on characteristics of the purchase, such as its size, time of usage, and restrictions.

In practice, road pricing is often nonlinear. For example, the tolls in Singapore (Menon et al., 1993), London (Santos and Shaffer, 2004), and Stockholm (Stockholmsforsoket, 2006) are not proportional to the distance traveled inside the tolling areas. In Stockholm, tolls are also not proportional to the number of times a user enters the tolling area. The amount of tolls paid on a given day is limited to SEK 60. After paying this maximum amount, users can freely enter the tolling area for the rest of the day. For its congestion charge, London offers monthly and annual passes to frequent users at an approximately 15% discount. Similarly, the Dulles Greenway's VIP Frequent Rider Program gives rebates to users with high mileage. During phase I of its Value Pricing Project on Interstate 15, San Diego sold $50 monthly permits that allow single occupancy vehicles to use lanes reserved for high occupancy vehicles. (During phase II, the permits were replaced by tolls.)

Despite its widespread use, the literature on nonlinear road pricing is limited. Our survey of the literature yielded no more than 10 journal articles addressing the topic. De Borger (2001) proposes a discrete choice model to study optimal two-part tariffs in the presence of externalities. Wang et al. (2011) considers three questions: (a) which nonlinear pricing scheme (among the five considered therein) is most profitable, (b) how does the most profitable choice depend on congestion, and (c) does usage-only pricing necessarily denominate other nonlinear schemes if congestion is severe? In both articles, the authors opine that nonlinear pricing has been overlooked in the literature. With an objective different from the previous two articles, Gabriel and Bernstein (1997a) presents a nonlinear complementarity problem (NCP) for finding a user equilibrium (UE) distribution on general road networks (or, more simply, the UE problem) when travel costs are not link-wise additive. In their formulation, one component of the path travel cost is a nonlinear function of its travel distance. To solve their UE problem, Gabriel and Bernstein (1997a) proposes an algorithm based on nonsmooth equations and sequential quadratic programming (see also Gabriel and Bernstein, 1997b). Lo and Chen (2000) considers a similar problem and converts their NCP into an unconstrained optimization problem based on a merit function. More recently, Agdeppa et al. (2007) modifies the model in Gabriel and Bernstein (1997a) by introducing a disutility function and formulate the problem as a monotone mixed complementarity problem instead. Maruyama and Harata (2006) and Maruyama and Sumalee (2007) propose an algorithm for area-based pricing, one form of nonlinear pricing. The authors of the last two papers observe that area-based pricing is not link-wise additive, and it may be intuitive to conclude that there exists no equilibrium condition based on link flows. As demonstrated in this report, this intuition is incorrect.

This report considers nonlinear pricing in the context of managing travel demand, reducing congestion, and, perhaps, lessening the environmental impact in a tolling area. Although it is common to assume that a tolling area consists of connected roads or roads in a connected geographical area, such an assumption is unnecessary. For example, a tolling area can consist of not necessarily connected roads or highways that are under the jurisdiction of a single entity (a government agency or private company). It is also possible to let the tolling area be the entire road network and every road user must pay tolls. Doing so reduces our problem to the one addressed in Gabriel and Bernstein (1997a).

To our knowledge, there has been little or no attempt to find an optimal nonlinear pricing scheme for a general road network. To find an optimal scheme, De Borger (2001) assumes that the travel demand is measured in kilometers without an explicit road network. Similarly, Wang et al. (2011) considers a network with only one link. In this report, we formulate the problem of finding a nonlinear pricing scheme that, e.g., maximizes the social benefit as a mathematical program with equilibrium constraints. We demonstrate that such a problem can be solved using a search algorithm when the tolling function is piecewise linear.

2 Fundamental Concepts

This section briefly describes key concepts such as user equilibrium, system optimum and tolled user equilibrium and explains their roles in road pricing.

In the literature, a road system is represented as a collection of nodes connected together with links or arcs such as the one shown in Figure 2-1 below.

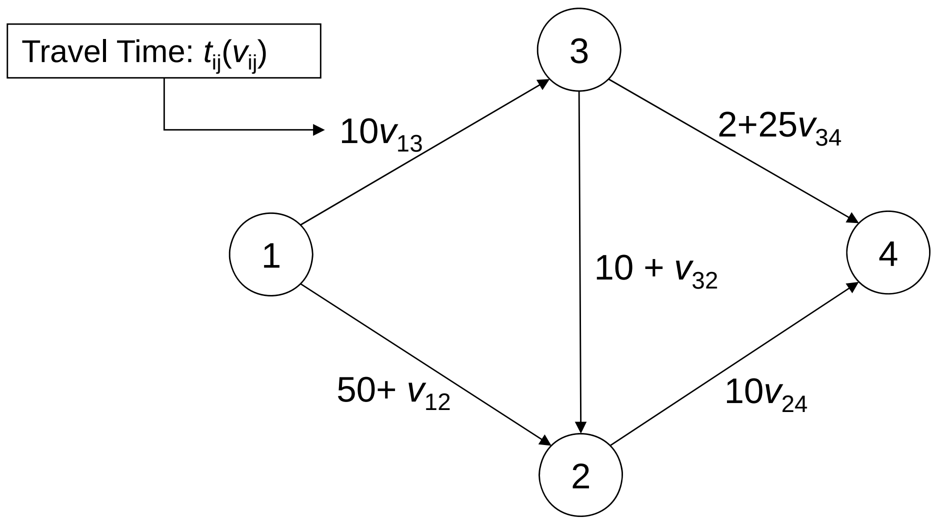


Figure 2-1: Four-node network

The network in Figure 2-1 has four nodes. It is sufficient to think of nodes as road intersections. (In general, nodes depending on applications can represent cities, highway entrances and exits, toll facilities, locations where, e.g., two lanes merge into one, etc.) Links denote roads that connect two intersections together. We refer to nodes by their numbers and a generic node is referred to simply by a letter i or j. A link or arc is represented as a pair of numbers, e.g., (3, 2) and a pair (i, j) denotes a generic link. The order of the two numbers in each link is important because it indicates the direction of the road. In the example, (3, 2) refers to the (one-way) road from node 3 to node 2. If the road between nodes 2 and 3 is two-way, then links, (3, 2) and (2, 3), would be present in Figure 2-1.

Next to each link is a function that provides the link's travel time. The value of this function is the time to traverse the link and this time depends on the number of cars on the link. Typically, more cars on the link mean longer travel times. In Figure 2-1, the travel time function for link (1, 3), or $t_{13}(v_{13})$, is $10v_{13}$, where v_{13} represents the number of cars per hour using arc (1, 3). (Below, we also refer to cars as (road) *users* and the number of cars on a link as *flow*[4].) If there are 2 cars on arc (1, 3), then $v_{13} = 2$ and the travel time for the arc is $10v_{13} = 10 \times 2 = 20$ minutes. Observe that the travel time would be longer than 20 minutes if v_{13} is larger than 2.

4 It is more accurate to use "flow rate" here. For simplicity, many in the literature use "flow" instead. Similarly, we should also use "cars per hour" instead of just cars.

(As written, the unit of flow equals one car. However, it is also possible to make one unit of flow equal 1000 cars and two units of flow on arc (1, 3) means 2000 cars. In this sense, it is meaningful to have fraction units of flow or, more intuitively, number of cars.)

In Figure 2-1, there are three paths from node 1 to node 4 and they are as shown in Table 2-1 below.

Table 2-1: Paths for the 4-node network

Path	
P1	$1 \rightarrow 3 \rightarrow 4$
P2	$1 \rightarrow 2 \rightarrow 4$
P3	$1 \rightarrow 3 \rightarrow 2 \rightarrow 4$

Assume that there are three cars traveling from node 1 to node 4. If each car uses a different path, then the number of cars on each link would be as shown on the network on the left of Figure 2-2. The network on the right of the figure gives the travel times computed from the formula in Figure 2-1. Using these times, the car using path P1 takes 47 (20 + 27) minutes to travel from node 1 to node 4. Similarly, the cars using P2 and P3 take 71 and 51 minutes, respectively. The total travel time or delay experienced by the three users/cars is 169 $(47 + 51 + 71)$ minutes. Among the three paths, path P1 is the shortest, thus most desirable. Cars using paths P2 and P3 would want to switch to path P1 instead. If they do, there would be three cars on path P1 and none on the other two. Consequently, there are three cars on arcs (1, 3) and (3, 4) and none on the other arcs. The corresponding travel times on arcs (1, 3) and (3, 4) are 30 and 77, respectively. This means that the travel time on path P1 is 107 minutes. On the other hand, arcs (1, 2), (3, 2) and (2, 4) have zero flow. As a result, the travel times for path P2 and P3 are 50 and 40, respectively. Thus, path P3 becomes desirable. Both cases (assigning one car to every path and assigning all three cars to one path, P1) lead to unstable situations because users have incentive to switch to a different route.

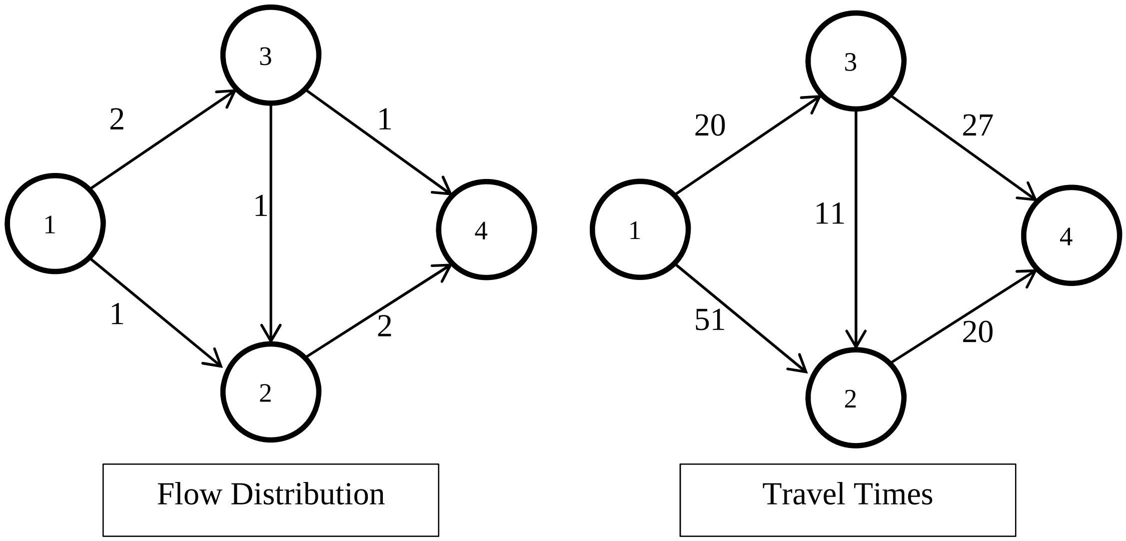


Figure 2-2: Link flows and travel times from assigning a unit flow to each of the three paths

2.1 User Equilibrium

Consider assigning 1.1389 cars (or units of flow) to P1 and 1.8611 to P3. Doing so yields the link and path flows and travel times displayed in Table 2-2. Observe that distributing the three units of flow among paths P1 and P3 as shown make the travel times on paths P1 and P3 the same (60.47). The travel time on P2 is higher at 68.61 minutes. Thus, P2 is undesirable and users of paths P1 and P3 would be unwilling to switch to P2. Because they have the same travel time, users of path P1 and P3 have no incentive to switch to the other path. So, the path and link flows in Table 2-2 are stable or in *user equilibrium* because no user has any incentive to switch to another route.

Table 2-2: User equilibrium flows and times

	Flow	**Travel Time**
P1: 1 → 3 → 4	1.1389	60.47
P2: 1 → 2 → 4	0.00	68.61
P3: 1 → 3 → 2 → 4	1.8611	60.47
Arc (1,3)	3.00	30.00
Arc (1,2)	0.00	50.00
Arc (3,2)	1.8611	11.86
Arc (3,4)	1.1389	30.47
Arc (2, 4)	1.8611	18.61

Although not trivial, the problem of determining the assignment of flows to paths in order to achieve user equilibrium can be formulated as an optimization problem that is relatively easy to solve. Most, if not all, commercial software packages for transportation planning have facilities to determine user equilibrium flow assignments. For our small example, we use the solver in EXCEL to determine the user equilibrium flows in Table 2-2: User equilibrium flows and times.

To add an additional feature, assume that there is a demand function that determines the number of users who want to travel from node 1 to node 4. For example, let the demand function for the origin-destination (OD) pair (1, 4) be $D_{14}(t_{14}) = 10 - 0.09007t_{14}$, where t_{14} is the travel time from node 1 to node 4. When $t_{14} = 0$, i.e., when travel is instantaneous, 10 users want to travel. As the travel time or the price of travel increases, the demand for travel decreases at the rate 0.09007 user per minute.

From Table 2-2, there are 3 users for OD pair (1, 4) and the travel time for these users is t_{14} = 60.47 minutes. When $t_{14} = 60.47$, the above demand function implies that there are 4.5529 (or $10 - 0.09007 \times 60.47$) users who want to travel. Similar to before, $t_{14} = 60.74$ is an unstable travel time because there are more than 3 users who want to travel at this travel time (or price of travel). However, when more users travel, the travel time will increase. Similar to before, allowing more users to travel may also lead to another unstable travel time for OD pair (1, 4).

Table 2-3 displays a solution that is in user equilibrium with respect to our demand function. In this table, there 3.6 (1.3222 + 2.7778) users and the travel time is 71.06 minutes when demand is distributed between paths P1 and P3 as shown. In addition, $D_{14}(71.06) = 10 - 0.09001 \times 71.06 = 3.6$. Thus, the solution in Table 2-3 is in user equilibrium for two reasons. One is that users of path P1 and P3 have no incentive to switch to another route. Switching to either P1 or P3 yields no gain while switching to P2 is worse. For the other reason, when the travel time is 71.06 minutes, the travel demand according to the demand function is 3.6. Thus, there is no shortage or excess demand. In other words, there is no incentive for users to change their demand for travel.

Table 2-3: User equilibrium flows and times under elastic demand

	Flow	Travel Time
P1: 1 → 3 → 4	1.3222	71.06
P2: 1 → 2 → 4	0	72.78
P3: 1 → 3 → 2 → 4	2.2778	71.06
Arc (1,3)	3.6	36.00
Arc (1,2)	0	50.00
Arc (3,2)	2.2778	12.28
Arc (3,4)	1.3222	35.06
Arc (2, 4)	2.2778	22.78

When travel demand is elastic or given by demand functions, the problem of determining a user equilibrium solution, as before, is an optimization problem relatively easy to solve. The solution in Table 2-3 is obtained using the solver in EXCEL.

2.2 System Optimum

Instead of finding a solution that is in user equilibrium, a system problem seeks a solution that maximizes the economic or social surplus. In economics, social surplus consists of consumer and producer surplus. Figure 3 graphically illustrates these two concepts. In our setting, the demand curve in the figure refers to the travel demand function in Section 2.1. The quantity and price on the x and y-axis represent to the travel demand and time (the price of travel). Consumer sur[lus is the area above the equilibrium travel time and below the demand curve. This reflects the fact that users would have been willing to travel at a travel time longer (i.e., pay a higher price) than the equilibrium travel time. The difference between the demand curve and the equilibrium travel time represents time-saving by consumers or users. Likewise, producer surplus is the area below the equilibrium travel time but above the supply curve, which describes how the equilibrium travel time varies with the number of users. When transportation agencies are viewed as producers of travel services, producer surplus reflects the fact that produces would have been willing to provide travel service to the first user at a price (time) lower than the equilibrium price, to the second user at a slightly higher price but still below the equilibrium price, etc. However, the produces receive the same equilibrium price for all the users who make the journey. Thus, the difference between equilibrium price and the supply curve represents the surplus or profit for the producer.

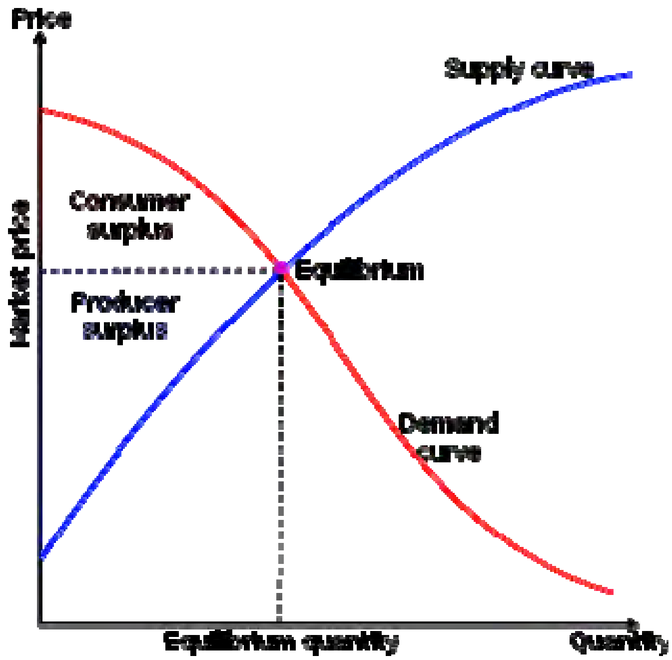


Figure 2-3: Consumer and producer surplus

Table 2-4 displays the system optimal solution using the data from the 4-node network. The social surplus for this solution is 117.85, an amount greater than the one for the equilibrium solution in Table 2-3 (71.94).

Table 2-4: System optimal flows and times under elastic demand

	Flow	Travel Time
P1: 1 → 3 → 4	0.8530	42.76
P2: 1 → 2 → 4	0.5322	66.76
P3: 1 → 3 → 2 → 4	1.0911	46.76
Arc (1,3)	1.9441	19.44
Arc (1,2)	0.5322	50.53
Arc (3,2)	1.0911	11.09
Arc (3,4)	0.8530	23.32
Arc (2, 4)	1.6233	16.23

2.3 Tolled User Equilibrium

When compared to the user equilibrium solution (Table 2-3), the system optimal solution in Table 2-4 generates less travel demand (2.4763 vs. 3.6) and less total travel time (123.04 vs. 255.80). This makes the system optimal solution more desirable because it makes the road network less crowded (less cars) and congested (less total travel time). As previously discussed, the system optimal solution is not stable in that travelers using paths P2 and P3 would want to switch to P1, a shorter path. On the other hand, we can charge tolls on links so as to discourage users from switching routes. Table 2-5 offers a set of tolls (measured in minutes[5]).

Table 2-5: Tolled user equilibrium solution

	Flow	**Travel Time**	**Tolls**	**Cost = time+tolls**
P1: 1 → 3 → 4	0.8530	42.76		83.53
P2: 1 → 2 → 4	0.5322	66.76		83.53
P3: 1 → 3 → 2 → 4	1.0911	46.76		83.53
Arc (1,3)	1.9441	19.44	19.44	38.88
Arc (1,2)	0.5322	50.53	0.53	51.06
Arc (3,2)	1.0911	11.09	1.09	12.18
Arc (3,4)	0.8530	23.32	21.32	44.64
Arc (2, 4)	1.6233	16.23	16.23	32.46

When added to the link travel times, the tolls in Table 2-5 make the travel cost of all three paths the same (83.53 minutes). Thus, no user has any incentive to change routes when tolls are added to the link travel times. Moreover, $D_{14}(83.53) = 2.4756$, essentially the same as the sum of the flows on the three paths in Table 2-5. When round-off errors are taken into account, this shows that the demand is also in equilibrium when tolls are present. In case, we refer to the solution Table 2-5 as a tolled user equilibrium solution.

5 Tolls measured in minutes can be converted into monetary values using the value of time. For example, a study indicates that the average value of time for a motorist in United Kingdom is £26.43 per hour.

3 Pricing Road Usage in an Area

Section 2 describes a form of road pricing that charges tolls on different links. This section discusses a form of road pricing that charges tolls or fees for using roads inside tolling areas. In practice, there is typically only one tolling area and it is often the central business district. At least three cities worldwide (Singapore, London and Stockholm) have success implementing this form of road pricing. For example, motorists in London have to pay a fixed rate of £8 per day to enter and use roads in the central business district. In Stockholm, motorists have to pay a toll each time they enter the tolling area. The amount of tolls paid on a given day is limited to SEK 60. When the maximum amount is reached, motorists can freely enter the area. Observe that tolls for London and Stockholm do not depend on the distance traveled inside the tolling area. This and subsequent sections consider tolls that depend on the distance traveled inside the tolling area instead. We do this for two reasons: (a) charging tolls based on distance traveled is a generalization of the road pricing in, e.g., London and, (b) when the tolling area is the entire road network, tolls based on distance reduces to the VMT (Vehicle-Miles Traveled) fee, the road usage fee that many think will replace the gas tax.

To illustrate our form of road pricing, consider the network in Figure 3-1. The shaded areas are tolling areas.

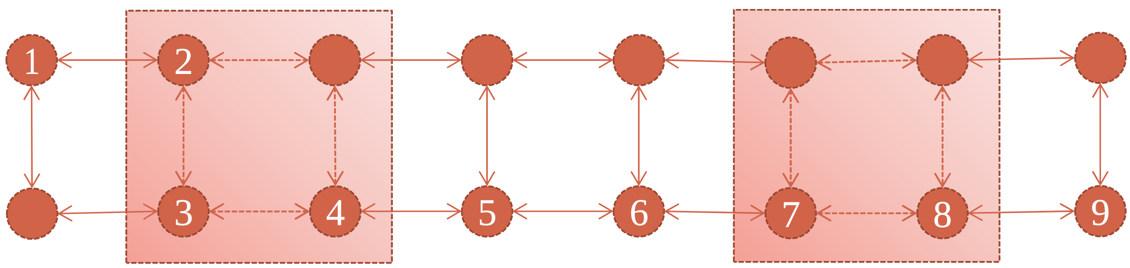


Figure 3-1: Tolling areas

Motorists traveling from node 1 to node 9 along the path $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9$ have to pay tolls based on the lengths of links (2, 3), (3, 4) and (7, 8).

3.1 Linear Road Pricing

In linear road pricing, the toll is a linear function of the distance traveled inside the tolling areas. Intuitively, linear road pricing would specify a fee and compute tolls from the distance traveled using this fee. In the above example, if the lengths of link (2, 3), (3, 4) and (7, 8) are 5, 8 and 3 miles, respectively, and the fee is \$0.5 per mile, then the toll associated with the above path from node 1 to node 9 is 0.5×16 or \$8. Mathematically, $T(\ell)$, the pricing function, is $\mu\ell$, where μ and ℓ are the fee and the distance traveled, respectively. For our example, $\mu = 0.5$ and $T(\ell) = 0.5\ell$. Moreover, $T(\ell)$ is linear road pricing because $\mu\ell$ is a linear function of ℓ. In fact, the VMT fee frequently discussed at meetings and conferences is another example of linear road pricing.

Mathematically, linear road pricing does not present any difficulty, e.g., in terms of finding a tolled user equilibrium solution because it can be treated as link tolls. The latter is discussed in Section 2. In particular, if we charge \$2.5 (0.5×5) toll on link (2, 3), \$4 (0.4×8) on link (3, 4), and \$1.5 (0.5×3) on link (7, 8), then the toll we collect from motorists using the path from 1 to 9 is the same as before. In the literature, many refer to linear road pricing as being link-wise additive, i.e., linear road pricing is simply the sum of tolls on individual links. Existing software packages for transportation planning can be used to analyze the effects of linear road pricing or VMT fees.

3.2 Nonlinear Road Pricing

In nonlinear road pricing, $T(\ell)$ is generally not a linear function of the distance traveled, ℓ. For example, if $T(\ell) = 10.2 + 0.04\ell^2$, then the toll associated with the above path from 1 to 9 is $10.2 + 0.04(16)^2$ or \$20.84. In our research, we focus on simple nonlinear functions because motorists may not understand and respond appropriately to complicated nonlinear functions. Below are functions that we considered in our research.

Figure 3-2 displays an example of two-part pricing and it is the graph of the tolling function $T(\ell) = 2.0 + 0.25\ell$. In words, $T(\ell)$ charges motorists \$2.0 to access tolling areas and \$0.25 per mile traveled inside these areas.

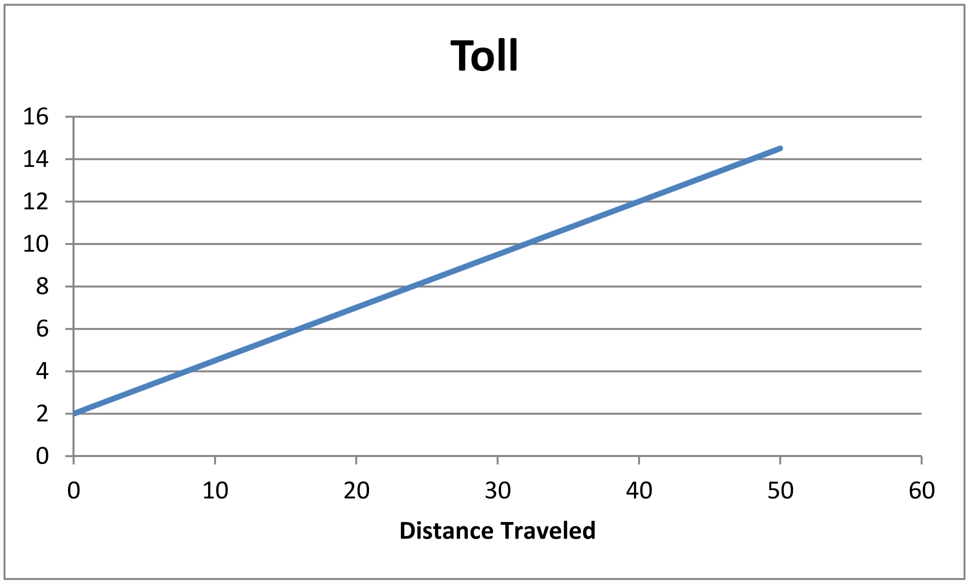


Figure 3-2: Two-part pricing

Figure 3-3 is a graph of the tolling function $T(\ell) = \max\{0.1\ell, 0.5\ell - 12\}$. Under this function, the VMT fee is $0.10 per mile if ℓ, the distance traveled, is no more than 30 miles. When ℓ is larger than 30 miles, the fee is $0.50 per mile traveled in excess of 30. The latter should discourage heavy road usage when unnecessary.

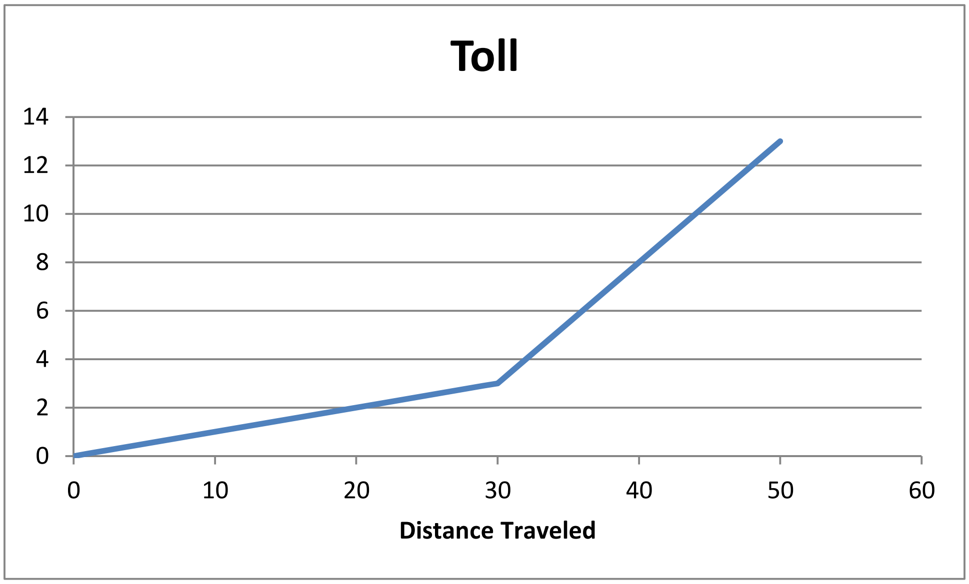


Figure 3-3: Pricing function for discouraging heavy road usage or 2-VMT pricing.

Figure 3-4 displays an example of three-part pricing and is a graph of the pricing function $T(\ell) = \max\{3.0, 0.2\ell\}$. Under this three-part pricing, motorists are charged $3 access fee. Associated with this fee, motorists can travel up to 15 miles for free. The travel distance exceeding 15 miles is charged at a rate of $0.20 per mile.

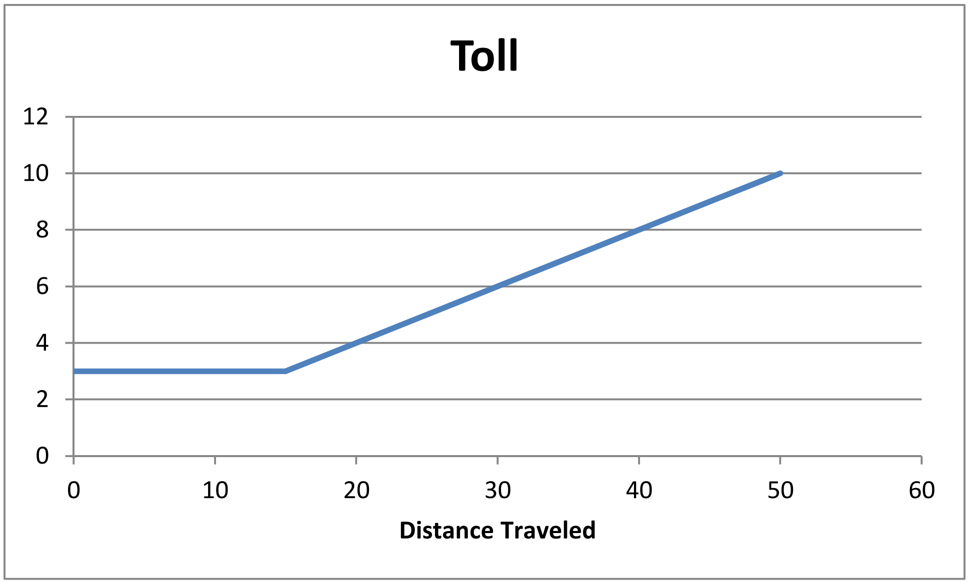


Figure 3-4: Three-part pricing

4 Socioeconomic Impacts of Road Pricing

Below we analyze the socioeconomic impacts of one-part and 2-VMT pricing (see Figure 3-3). (Recall that many also refer to one-part pricing as flat mileage or VMT fee.) The analysis in this section is at a macro level because of the available data. The purpose of our analysis is to demonstrate a potential application of nonlinear pricing.

4.1 Data

The data for our analysis are from the 2009 National Household Travel Survey (NHTS) that consists of four files: household (HH), person, trip and vehicle. In these files, 15,884, 30,952, 11,491 and 29,457 entries, respectively, are from Florida. Because our analysis was performed at the HH level, some attributes from the vehicle and person file are merged with entries in the HH file. After removing incomplete and/or meaningless entries, there are 13,086 entries remaining. Our analyses below address these entries and we refer to them as the *clean* HH data.

Tables 4-1 to 4-7 summarize the clean HH data. Table 4-1 indicates that, on average, each HH in rural and urban areas own approximately two vehicles and they also have similar fuel efficiencies (as measured in miles per gallon or MPG). Based on the annual vehicle miles traveled (VMT), rural HH travel more than those in urban areas do. Interestingly, Table 4-4 shows that those with higher income travel more than those with less. Tables 4-2 and 4-3 display types of vehicles (cars, vans, sport utility vehicles or SUV, trucks, recreational vehicles or RV, and motor cycles) owned by HH in Florida. In urban areas, there is a higher percentage of car ownerships, 55.17% versus 42.69% in rural areas. As indicated in Table 4-6, the percentage of SUV ownerships is also more among HH with incomes higher than $40,000. Finally, Table 4-7 reports the average fuel efficiency and annual VMT for HH in each county.

Table 4-1: Descriptive statistics by location

	No. HH (Percent HH)	Tot. Annual Income per HH in $	No. Veh. per HH	Avg. Veh. MPG per HH	Annual VMT per HH
Rural	2775 (21.2)	59984	2.2	20.79	25552
Urban	10311 (78.8)	63706	1.88	21.40	19905
Overall Avg.	-	62917	1.93	21.27	21056

Table 4-2: Number of vehicles by type and location

	Car	Van	SUV	Truck	RV	Motor Cycle	Total
Rural	2506	471	1104	1512	46	231	5870
Urban	10666	1609	3792	2558	132	577	19334

Table 4-3: Percent of vehicles by type and location

	Car	Van	SUV	Truck	RV	Motor Cycle	Total
Rural	42.69	8.02	18.81	25.76	0.78	3.94	100
Urban	55.17	8.32	19.61	13.23	0.68	2.98	100

Table 4-4: Descriptive statistics by income group

Income Group	Household			No. Veh. per HH	Avg. Veh. MPG per HH	Tot. Annual VMT per HH
	Total HH	Rural HH (% HH)	Urban HH (% HH)			
$0-$19,999	2119	475 (22.4)	1644 (77.6)	1.4	20.64	12187
$20,000-$39,999	3288	737 (22.4)	2551 (77.6)	1.64	21.04	15926
$40,000-$59,9999	2468	537 (21.8)	1931 (78.2)	1.91	21.49	21035
$60,000-$79,999	1800	373 (20.7)	1427 (79.3)	2.16	21.78	24650
$80,000 to $200,000	3411	653 (19.1)	2758 (80.9)	2.42	21.46	29629

Table 4-5: Number of vehicles by type and income group

Income Group	Car	Van	SUV	Truck	RV	Motor Cycle	Total
0-19,999	1786	276	342	492	16	59	2971
20,000-39,999	2957	513	814	922	38	141	5385
40,000-59,9999	2459	391	854	835	40	142	4721
60,000-79,999	1935	305	800	664	31	152	3887
80,000 to 200,000	4035	595	2086	1157	53	314	8240
Total	13172	2080	4896	4070	178	808	25204

Table 4-6: Percent of vehicles by type and income group

Income Group	Car	Van	SUV	Truck	RV	Motor Cycle	Total
0-19,999	60.11	9.29	11.51	16.56	0.54	1.99	100
20,000-39,999	54.91	9.53	15.12	17.12	0.71	2.62	100
40,000-59,9999	52.09	8.28	18.09	17.69	0.85	3.01	100
60,000-79,999	49.78	7.85	20.58	17.08	0.80	3.91	100
80,000 to 200,000	48.97	7.22	25.32	14.04	0.64	3.81	100

Table 4-7: Descriptive statistics by county

Sl. No.	County Name	No. of HH	Avg. MPG	Avg. VMT	Sl. No.	County Name	No. of HH	Avg. MPG	Avg. VMT
1	Alachua	203	21.74	22215	35	Lee	420	21.08	19261
2	Baker	35	19.6	28692	36	Leon	274	21.56	25573
3	Bay	157	20.27	24143	37	Levy	65	21.08	32177
4	Bradford	42	20.37	31243	38	Liberty	5	19.82	31286
5	Brevard	373	21.56	18338	39	Madison	28	19.67	21641
6	Broward	1091	22.13	20500	40	Manatee	180	20.75	16495
7	Calhoun	20	20.14	28195	41	Marion	257	20.94	19098
8	Charlotte	129	22.1	20010	42	Martin	117	20.94	22000
9	Citrus	299	20.88	21920	43	Miami-dade	1256	21.74	19607
10	Clay	175	20.92	25895	44	Monroe	170	20.8	21281
11	Collier	209	21.17	18020	45	Nassau	67	21.71	28517
12	Columbia	66	21.25	26898	46	Okaloosa	182	20.92	24101
13	DeSoto	44	19.54	19390	47	Okeechobee	51	20.1	19154
14	Dixie	21	18.74	32041	48	Orange	432	21.92	23271
15	Duval	523	21.97	22077	49	Osceola	104	21.5	25155
16	Escambia	280	21.31	22062	50	Palm Beach	815	21.92	18712
17	Flagler	124	23.05	24678	51	Pasco	309	20.75	19315
18	Franklin	24	18.62	20832	52	Pinellas	690	21.32	15846
19	Gadsden	35	21.59	24346	53	Polk	340	21.23	19687
20	Gilcrist	25	20.54	48320	54	Putnam	129	20.87	26888
21	Glades	15	19.36	25541	55	St. Johns	178	21.33	26101
22	Gulf	29	19.72	23260	56	St. Lucie	235	21.45	19940
23	Hamilton	25	19.4	27880	57	Santa Rosa	170	20.67	25298
24	Hardee	22	20.98	27697	58	Sarasota	249	21.95	19347
25	Hendry	38	20.44	23109	59	Seminole	250	21.19	20663
26	Hernando	109	21.77	20598	60	Sumter	62	21.33	16216
27	Highlands	224	20.5	18780	61	Suwannee	92	19.99	26018
28	Hillsborough	657	21.42	21346	62	Taylor	32	19.4	41091
29	Holmes	35	20.71	24735	63	Union	14	18.93	24759
30	Indian River	94	20.64	23064	64	Volusia	384	21.38	18281
31	Jackson	69	19.7	28926	65	Wakulla	21	21.83	32001
32	Jefferson	35	21.24	22825	66	Walton	62	21.77	25177
33	Lafayette	9	18.49	23346	67	Washington	41	21.26	23709
34	Lake	169	21.18	20774					

4.2 Methodology

Our analysis requires estimates of travel demands for various HH and one measure of such demands is the annual VMT. Below is a log-log linear regression model that can provide such estimates. The model assumes that the travel demand of a HH is a function of its social characteristics and the available transportation services. Mathematically, the model can be written as follows:

$$\begin{aligned} \ln(M) = \beta_0 + \beta_1 \ln(PM) & \\ & + \beta_2 \ln(hhtotinc) \\ & + \beta_3 \ln(hhvehcnt) + \beta_4 U \\ & + \beta_5 \ln(hhtotinc) \ln(PM) + \beta_6 SUB \\ & + \beta_7 \ln(PM) SUB + \beta_8 wrkcnt + \beta_9 hhchild \end{aligned} \qquad (4.1)$$

where,

M = The annual VMT of a household

PM = Weighted average of fuel cost per mile for a household, calculated as follows: $= \sum_i PM_i \times M_i / \sum_i M_i$, where $PM_i = \frac{NetGasPrice+StateGasTax}{MPG_i} + VMTFee$ (i represents individual vehicles in a household) . When VMTFee = 0, PM_GasTax = PM. Similarly, when *StateGasTax*=0, *PM_VMTFee= PM*,

hhtotinc= total annual household income

hhvehcnt= household vehicle count

U= 1 if household is located in urban area, otherwise 0

SUB = 1 if household has different types of vehicles among car, van, SUV, truck and RV; 0 otherwise.

wkrcnt= number of workers in a household

hhchild= number of children in a household

In Eq. (4.1), the values for the independent variables (e.g., *PM* , *hhtotinc*, *U*, *SUB*, etc.) are from the clean HH data discussed earlier and, as in linear regression models, those for $\beta's$ are chosen to minimize the sum of squared errors. Doing so yields the adjusted R-square of 0.56. As shown in Table 4-8, all $\beta's$ have the correct sign and are statistically significant. The resulting demand elasticity for each income group is listed in Table 4-9. The elasticity is calculated as follows:

$$e = \frac{(\Delta M / M)}{(\Delta PM / PM)} = \frac{\delta M \cdot PM}{\delta PM \cdot M} = \beta_1 + \beta_5 \ln(hhtotinc) + \beta_7 SUB \qquad (4.2)$$

Table 4-8: Estimated model

Independent factor	$\beta's$	Std. Error	*t*-statistic
Constant	β_0 =-2.4787	0.6472	-3.8298
ln(PM)	β_1 =-5.4067	0.3416	-15.8274
ln(hhtotinc)	β_2 =0.7612	0.0620	12.2745
ln(hhvehcnt)	β_3 =0.9188	0.0196	46.9485
U	β_4 =-0.1821	0.0147	-12.3627
ln(hhtotinc)*ln(PM)	β_5 =0.3449	0.0327	10.5518
SUB	β_6 =1.6881	0.1163	14.5149
ln(PM)*SUB	β_7 =0.7499	0.0607	12.3453
wrkcnt	β_8 =0.1509	0.0084	18.0693
hhchild	β_9 =0.1023	0.0079	12.8797

Table 4-9: Elasticity by income group based on average income

Income Group	Avg. Income ($)	Elasticity with SUB	Elasticity without SUB
$0 - $19,999	10000	-1.40	-2.15
$20,000 - $39,999	30000	-1.10	-1.85
$40,000 - $59,9999	50000	-0.92	-1.67
$60,000 - $79,999	70000	-0.80	-1.55
$80,000 - $200,000	140000	-0.59	-1.34

With the estimated model, the changes in consumer surplus (CS), revenue and social welfare (SW) can be estimated as follows:

$$\begin{gathered}\Delta CS = 0.5 \times (PM\_GasTax - PM\_VMTFee) \times (Miles\_GasTax + Miles\_VMTFee) \\ \Delta \text{Revenue} = VMTFee \times Miles\_VMTFee - (StateGasTax / AvgMPG) \times Miles\_GasTax \\ \Delta SW = \Delta CS + \Delta Revenue\end{gathered} \tag{4.3}$$

where,

PM_GasTax= *PM* under gasoline tax
PM_VMTFee= *PM* under mileage fee
Miles_GasTax= Annual total mile driven by a household under gasoline tax
Miles_VMTFee=Annual total mile driven by a household under mileage fee
ΔCS= Change in CS
ΔRevenue= Change in Revenue
ΔSW= Change in SW

4.3 One-Part Pricing

The current federal tax for gasoline is 18.4 cents/gallon. In Florida, the combined state and county tax for gasoline is 34.5 cents/gallon on average. (For convenience, we refer to this combination of state and county tax for gasoline simply as the state gas tax, where "state" refers to Florida in this report.) Using 21 MPG as the average fuel efficiency for vehicles in Florida, the current state gas tax of 34.5 cents per gallon is equivalent to 1.64 cents per mile. This calculation does not consider the travel behavior changes due to the change in the gas price. To obtain a revenue-neutral impact fee and fees for other purposes, we used Eq. (4.3) to calculate the changes in CS, revenue and SW and Table 4-10 displays the results from these calculations.

Table 4-10: Changes due to different mileage fees

Mileage fee (cents/mile)	Total change in CS ($)	Total change in Revenue ($)	Total change in SW ($)	% VMT reduction
1.60	151780	-16177	135602	0.49
1.61	127431	5030	132461	0.57
1.62	103102	26204	129306	0.65
1.63	78791	47345	126136	0.73
1.64	54500	68452	122953	0.81
1.70	-90849	194409	103560	1.30
2.00	-807692	807027	-666	3.70
2.20	-1276853	1200298	-76555	5.29
2.80	-2646108	2313856	-332252	10.04
4.10	-5445869	4437238	-1008631	20.14

As shown in Table 4-10, a flat mileage fee of 1.61 cents/mile is sufficient to maintain the current revenue level (without considering the difference in the operation cost). The mileage fees of 2.8 and 4.1 cents/mile can reduce annual vehicle miles traveled by approximately 10 and 20 percent, respectively. In the following, we present the distributional impacts of 1.61, 2.8, and 4.1 cents/mile mileage fees across different income groups and counties.

4.3.1 Mileage Fee of 1.61 Cents/Mile (Revenue-Neutral Fee)

Table 4-11 to Table 4-13 and Figure 4-1 summarize the impacts of $1.61 mileage fee. In Table 4-11, the average change in CS as a percentage of the average income is negligible in all income groups. Similarly, the changes in CS in Table 4-12 are also relatively insignificant for both rural and urban areas. In absolute terms, individuals living in rural areas receive higher benefits than those in the urban areas. Across different counties, the average change in CS ranges from -13.92 to 77.93 (Table 4-13). Other than Flagler, Hernando, Broward and Charlotte counties, the average changes in CS are positive. Due to a better fuel efficiency of vehicles, residents of these counties are slightly worse off. On the other hand, those in counties with less efficient vehicles benefit from the new policy.

Table 4-11: Average changes by income group with revenue-neutral fee

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	2.18	0.02	0.49	2.67
20,000-39,999	3288	4.47	0.01	1.12	5.59
40,000-59,9999	2468	5.94	0.01	2.99	8.94
60,000-79,999	1800	8.62	0.01	5.24	13.86
80,000 to 200,000	3411	22.85	0.02	-4.84	18.01

Table 4-12: Average changes by location with revenue-neutral fee

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	22.30	-5.73	16.57
Urban	10311	6.36	2.03	8.39

Table 4-13: Average changes by county with revenue-neutral fee

Sl. No.	County Name	No. of HH	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in Welfare ($)
1	Alachua	203	10.75	4.47	15.22
2	Baker	35	54.44	-30.51	23.94
3	Bay	157	25.27	-12.34	12.92
4	Bradford	42	25.52	-0.54	24.98
5	Brevard	373	2.89	6.39	9.27
6	Broward	1091	-1.46	7.92	6.46
7	Calhoun	20	32.15	-19.49	12.66
8	Charlotte	129	-1.04	8.68	7.64
9	Citrus	299	13.36	-2.36	11.00
10	Clay	175	13.75	-1.76	12.00
11	Collier	209	11.99	-1.37	10.62
12	Columbia	66	15.59	-3.41	12.18
14	DeSoto	44	33.99	-20.55	13.44
15	Dixie	21	77.93	-47.34	30.59
16	Duval	523	4.67	5.00	9.67
17	Escambia	280	10.52	2.47	12.99
18	Flagler	124	-13.92	24.14	10.21
19	Franklin	24	51.43	-29.89	21.54
20	Gadsden	35	0.13	8.27	8.40
21	Gilcrist	25	74.57	-33.37	41.20
22	Glades	15	35.10	-19.52	15.57
23	Gulf	29	39.75	-30.35	9.40
24	Hamilton	25	31.18	-17.34	13.84
25	Hardee	22	27.27	-9.53	17.74
26	Hendry	38	31.18	-10.54	20.64
27	Hernando	109	-4.78	13.59	8.80
28	Highlands	224	20.20	-10.99	9.21
29	Hillsborough	657	13.18	-3.27	9.92
30	Holmes	35	14.88	-5.72	9.16
31	Indian River	94	20.50	-5.87	14.63
32	Jackson	69	42.87	-20.13	22.74
33	Jefferson	35	1.28	9.52	10.79
34	Lafayette	9	61.30	-40.97	20.34
35	Lake	169	6.92	1.33	8.26
36	Lee	420	11.75	-2.55	9.20
37	Leon	274	10.80	3.27	14.07
38	Levy	65	20.91	-0.09	20.82
39	Liberty	5	40.92	-23.11	17.81

40	Madison	28	41.71	-24.43	17.27
41	Manatee	180	11.95	-4.16	7.79
42	Marion	257	9.67	-2.70	6.97
43	Martin	117	14.14	-3.95	10.19
44	Miami-dade	1256	1.12	4.78	5.90
45	Monroe	170	27.72	-7.45	20.27
46	Nassau	67	13.61	-0.54	13.07
47	Okaloosa	182	29.96	-12.76	17.21
48	Okeechobee	51	25.52	-9.40	16.12
49	Orange	432	2.78	5.10	7.88
50	Osceola	104	19.46	-5.53	13.93
51	Palm Beach	815	1.35	5.14	6.50
52	Pasco	309	10.83	-2.46	8.37
53	Pinellas	690	5.89	-0.33	5.56
54	Polk	340	11.44	-1.84	9.60
55	Putnam	129	12.33	-1.32	11.01
56	St. Johns	178	19.65	-2.74	16.90
57	St. Lucie	235	4.93	5.04	9.97
58	Santa Rosa	170	28.40	-11.24	17.16
59	Sarasota	249	8.73	2.60	11.32
60	Seminole	250	13.37	-2.32	11.06
61	Sumter	62	4.33	3.62	7.95
62	Suwannee	92	29.09	-8.98	20.11
63	Taylor	32	59.73	-43.39	16.35
64	Union	14	67.21	-41.73	25.48
65	Volusia	384	7.22	3.21	10.43
66	Wakulla	21	39.15	6.93	46.08
67	Walton	62	18.39	9.70	28.09
68	Washington	41	23.05	-5.43	17.63

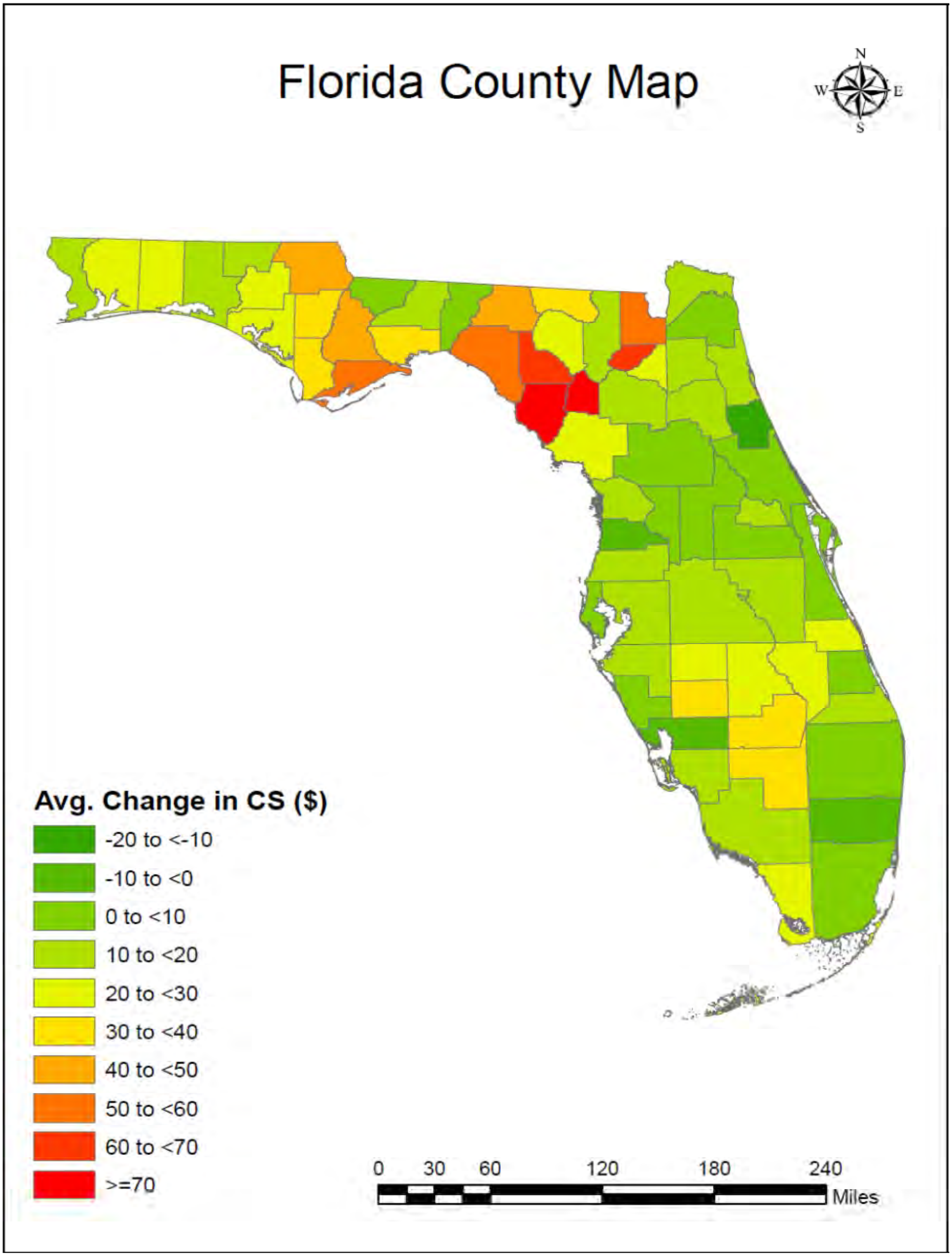


Figure 4-1: Spatial distribution of impacts of mileage fee of 1.61 cents/mile

4.3.2 Mileage Fee of 2.8 Cents/Mile (10% VMT Reduction)

The impacts of flat mileage fee of 2.8 cents/mile are presented in Table 4-14 to Table 4-16 and Figure 4-2. Such a high fee would lead to negative changes in CS for all income group and locations. The impacts are also regressive, i.e., the lower-income people suffer more than those with higher income. Individuals in the rural areas suffer more than those in the urban areas. The distributional impacts are presented in Figure 4-2. The dispersion of the impacts yields a Gini coefficient of 0.06, suggesting that the spatial equity is not a big concern.

Table 4-14: Average changes by income group 10% VMT Reduction

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-106.43	-0.84	77.84	-28.59
20,000-39,999	3288	-149.67	-0.49	120.95	-28.72
40,000-59,9999	2468	-200.53	-0.40	171.68	-28.84
60,000-79,999	1800	-246.95	-0.35	221.42	-25.53
80,000 to 200,000	3411	-289.97	-0.22	272.35	-17.62

Table 4-15: Average changes by location 10% VMT Reduction

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-238.21	214.76	-23.44
Urban	10311	-192.52	166.61	-25.91

Table 4-16: Average changes by county 10% VMT Reduction

Sl. No.	County Name	No. of HH	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in Welfare ($)
1	Alachua	203	-258.80	232.58	-26.28
2	Baker	35	-222.00	208.79	-12.33
3	Bay	157	-186.40	167.44	-19.01
4	Bradford	42	-267.90	252.05	-15.87
5	Brevard	373	-198.50	172.90	-25.69
6	Broward	1091	-215.30	184.28	-31.06
7	Calhoun	20	-200.50	179.42	-21.08
8	Charlotte	129	-201.40	170.42	-31.04
9	Citrus	299	-189.00	163.24	-25.80
10	Clay	175	-230.90	206.54	-24.42

11	Collier	209	-184.50	162.80	-21.75
12	Columbia	66	-211.60	185.21	-26.43
14	DeSoto	44	-146.20	129.10	-17.19
15	Dixie	21	-211.00	200.72	-10.28
16	Duval	523	-224.30	196.19	-28.12
17	Escambia	280	-209.70	187.48	-22.30
18	Flagler	124	-252.60	220.78	-31.84
19	Franklin	24	-188.30	176.67	-11.63
20	Gadsden	35	-228.80	193.36	-35.45
21	Gilcrist	25	-169.10	175.67	6.48
22	Glades	15	-182.90	169.29	-13.68
23	Gulf	29	-161.20	136.43	-24.78
24	Hamilton	25	-204.20	183.20	-20.02
25	Hardee	22	-216.90	195.84	-21.14
26	Hendry	38	-189.90	175.77	-14.20
27	Hernando	109	-195.80	166.49	-29.31
28	Highlands	224	-150.00	129.31	-20.75
29	Hillsborough	657	-201.50	177.10	-24.42
30	Holmes	35	-190.10	161.52	-28.66
31	Indian River	94	-181.10	163.99	-17.20
32	Jackson	69	-222.00	208.89	-13.15
33	Jefferson	35	-248.50	221.49	-27.04
34	Lafayette	9	-179.60	171.55	-8.12
35	Lake	169	-176.20	151.59	-24.63
36	Lee	420	-183.60	160.13	-23.48
37	Leon	274	-241.80	220.20	-21.63
38	Levy	65	-275.50	247.03	-28.55
39	Liberty	5	-216.00	194.12	-21.94
40	Madison	28	-170.30	154.04	-16.29
41	Manatee	180	-162.70	141.34	-21.37
42	Marion	257	-172.70	145.05	-27.69
43	Martin	117	-197.60	175.08	-22.53
44	Miami-dade	1256	-207.00	175.62	-31.43
45	Monroe	170	-189.20	178.59	-10.70
46	Nassau	67	-242.60	215.94	-26.76
47	Okaloosa	182	-226.80	208.67	-18.16
48	Okeechobee	51	-157.40	144.12	-13.30
49	Orange	432	-230.20	199.29	-30.97
50	Osceola	104	-237.80	210.65	-27.24
51	Palm Beach	815	-187.80	160.32	-27.53
52	Pasco	309	-185.90	159.79	-26.12

53	Pinellas	690	-165.50	139.95	-25.60
54	Polk	340	-189.10	163.02	-26.10
55	Putnam	129	-223.10	196.71	-26.47
56	St. Johns	178	-238.90	219.93	-19.00
57	St. Lucie	235	-190.90	168.45	-22.46
58	Santa Rosa	170	-236.20	216.10	-20.10
59	Sarasota	249	-189.20	166.23	-23.01
60	Seminole	250	-212.70	188.64	-24.09
61	Sumter	62	-140.00	117.46	-22.54
62	Suwannee	92	-223.40	203.83	-19.66
63	Taylor	32	-227.20	208.90	-18.33
64	Union	14	-214.70	205.59	-9.19
65	Volusia	384	-180.30	156.20	-24.10
66	Wakulla	21	-269.00	269.43	0.42
67	Walton	62	-245.00	233.37	-11.68
68	Washington	41	-234.00	204.99	-29.20

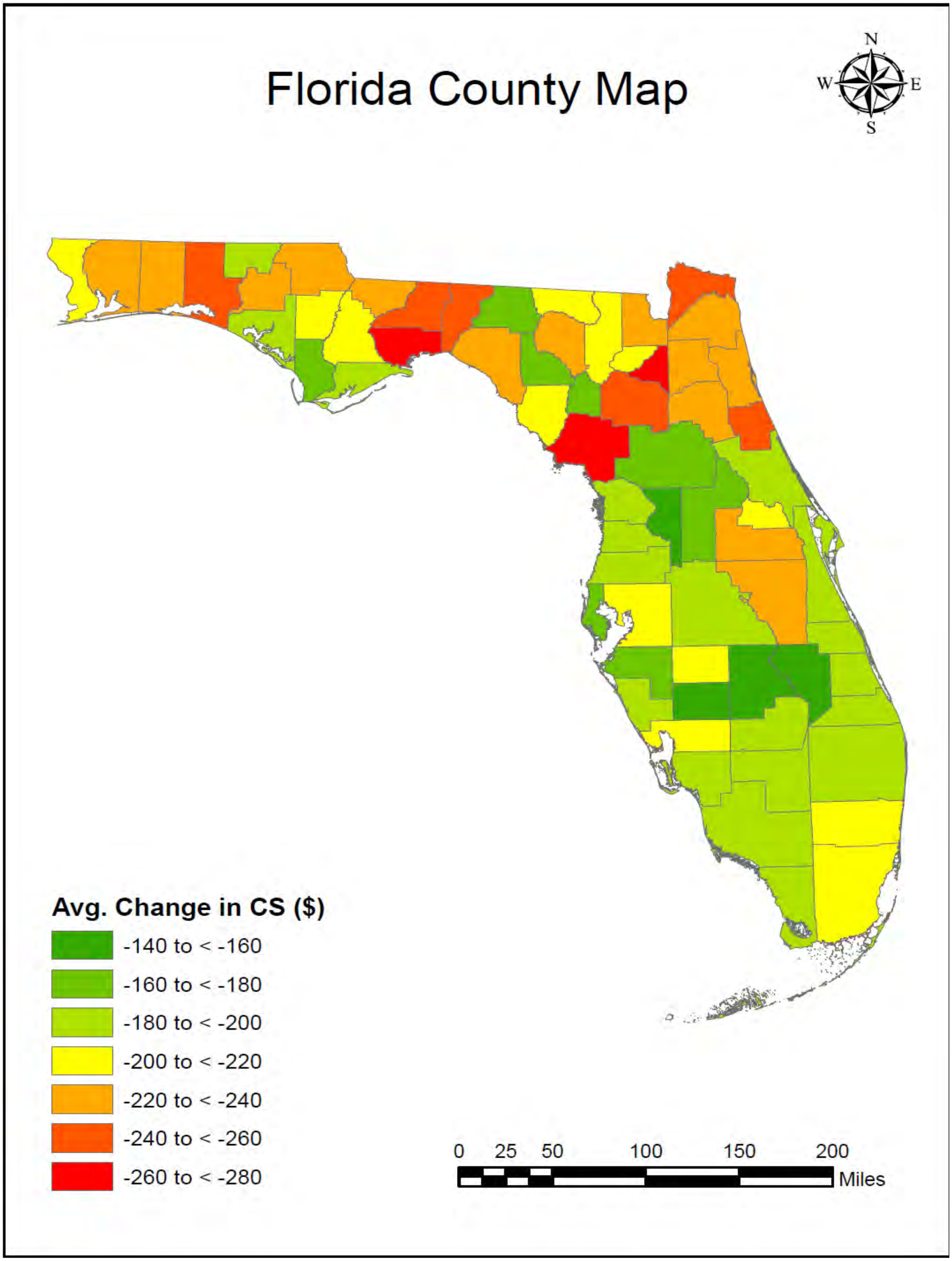


Figure 4-2: Spatial distribution of impacts of mileage fee of 2.8 cents/mile

4.3.3 Mileage Fee of 4.1 Cents/Mile (20% VMT Reduction)

The impacts of Flat fee of 4.1 cents/mile are presented in Table 4-17 to Table 4-19 and Figure 4-3: Spatial distribution of impacts of mileage fee of 4.1 cents/mile. The impacts among different income groups are similar to those from 2.8 cents/mile fee, but are more regressive. The rural residents again suffer more than those in urban areas do. However, the spatial distribution of the impacts is relatively uniform, yielding the same Gini coefficient of 0.06.

Table 4-17: Average changes by income group with *20% VMT reduction*

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-210.52	-1.66	138.34	-72.17
20,000-39,999	3288	-301.51	-1.00	223.69	-77.82
40,000-59,9999	2468	-407.39	-0.81	323.56	-83.84
60,000-79,999	1800	-506.37	-0.71	422.81	-83.55
80,000 to 200,000	3411	-613.16	-0.47	542.07	-71.10

Table 4-18: Average changes by location with *20% VMT reduction*

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-502.54	420.32	-82.22
Urban	10311	-392.91	317.22	-75.69

Table 4-19: Average changes by county with *20% VMT reduction*

Sl. No.	County Name	No. of HH	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in Welfare ($)
1	Alachua	203	-532.60	445.50	-87.10
2	Baker	35	-502.90	436.29	-66.62
3	Bay	157	-401.30	335.30	-66.14
4	Bradford	42	-567.00	490.27	-76.79
5	Brevard	373	-401.40	324.93	-76.50
6	Broward	1091	-430.60	345.41	-85.25
7	Calhoun	20	-437.10	366.23	-70.95
8	Charlotte	129	-402.90	317.04	-85.93
9	Citrus	299	-392.00	312.99	-79.08
10	Clay	175	-479.70	401.70	-78.06
11	Collier	209	-383.00	314.05	-69.00

12	Columbia	66	-440.90	358.41	-82.54
14	DeSoto	44	-327.50	265.56	-61.96
15	Dixie	21	-504.70	433.48	-71.30
16	Duval	523	-456.00	373.04	-82.97
17	Escambia	280	-432.60	358.54	-74.08
18	Flagler	124	-492.80	400.01	-92.86
19	Franklin	24	-432.40	371.10	-61.38
20	Gadsden	35	-457.50	358.89	-98.64
21	Gilcrist	25	-417.60	373.01	-44.59
22	Glades	15	-405.60	348.32	-57.36
23	Gulf	29	-363.40	288.83	-74.61
24	Hamilton	25	-443.50	373.04	-70.50
25	Hardee	22	-463.70	385.19	-78.60
26	Hendry	38	-413.80	348.46	-65.39
27	Hernando	109	-386.60	302.76	-83.88
28	Highlands	224	-321.00	256.69	-64.32
29	Hillsborough	657	-418.90	344.45	-74.53
30	Holmes	35	-395.30	311.96	-83.37
31	Indian River	94	-385.10	321.17	-64.02
32	Jackson	69	-492.50	425.76	-66.77
33	Jefferson	35	-502.30	419.54	-82.79
34	Lafayette	9	-427.50	376.72	-50.84
35	Lake	169	-359.80	287.35	-72.48
36	Lee	420	-380.70	309.57	-71.15
37	Leon	274	-499.60	425.23	-74.38
38	Levy	65	-574.30	473.14	-101.10
39	Liberty	5	-475.60	394.32	-81.30
40	Madison	28	-384.90	319.66	-65.27
41	Manatee	180	-338.90	275.02	-63.89
42	Marion	257	-355.00	277.34	-77.71
43	Martin	117	-412.40	341.91	-70.54
44	Miami-dade	1256	-416.30	331.02	-85.32
45	Monroe	170	-410.60	354.32	-56.34
46	Nassau	67	-503.00	418.21	-84.80
47	Okaloosa	182	-489.20	418.82	-70.47
48	Okeechobee	51	-342.00	285.45	-56.58
49	Orange	432	-465.60	378.16	-87.49
50	Osceola	104	-498.50	411.27	-87.30
51	Palm Beach	815	-377.90	301.20	-76.72
52	Pasco	309	-383.90	307.88	-76.05
53	Pinellas	690	-337.50	266.93	-70.64

54	Polk	340	-390.70	313.02	-77.75
55	Putnam	129	-461.60	380.19	-81.42
56	St. Johns	178	-503.20	431.18	-72.02
57	St. Lucie	235	-388.50	318.63	-69.91
58	Santa Rosa	170	-506.10	430.94	-75.22
59	Sarasota	249	-388.50	315.64	-72.92
60	Seminole	250	-442.10	366.59	-75.53
61	Sumter	62	-282.60	216.10	-66.50
62	Suwannee	92	-479.00	400.88	-78.19
63	Taylor	32	-522.20	451.91	-70.33
64	Union	14	-504.30	442.99	-61.33
65	Volusia	384	-368.10	294.12	-74.07
66	Wakulla	21	-582.30	515.13	-67.17
67	Walton	62	-512.30	441.71	-70.65
68	Washington	41	-492.00	395.52	-96.52

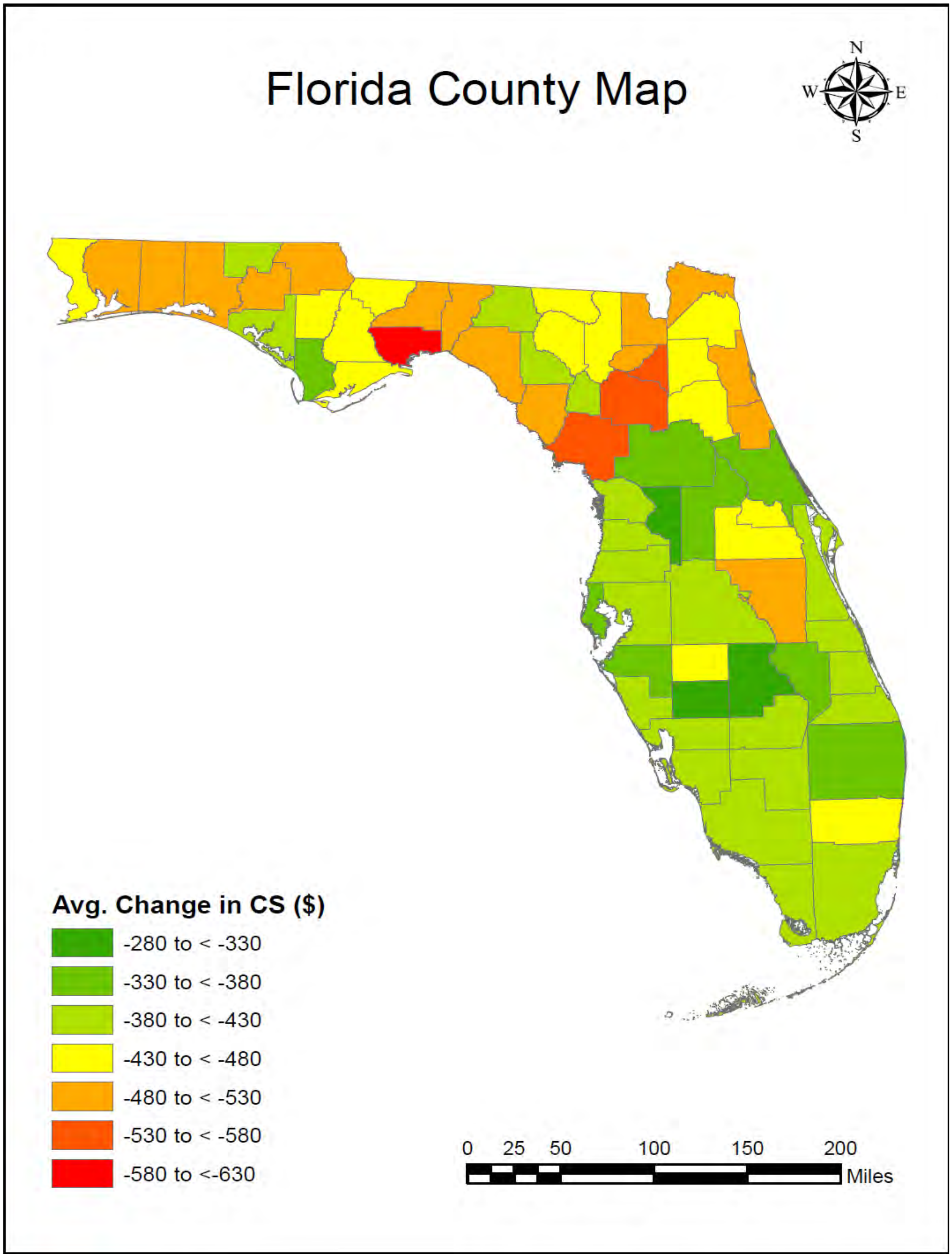


Figure 4-3: Spatial distribution of impacts of mileage fee of 4.1 cents/mile

4.4 Two-VMT Pricing

Although they encourage less driving, the above analysis reveals that high mileage fees, in particular those that are 10% and 20% more than the revenue-neutral fee, are regressive. Below, we consider two-VMT pricing schemes in an effort to lessen the degree of regressivity while encouraging individuals to travel less and, consequently, reducing traffic congestion. In a two-VMT pricing scheme, vehicles are charged at a lower mileage fee if the distance traveled is no more than a specified threshold distance. Each mile traveled beyond the threshold is charged at a higher fee. The motivation for this scheme follows from the statistics in Section 4.1, which indicate that, on average, individuals with low income travel less than those with higher income.

4.4.1 Two-VMT Pricing vs. Flat Fee of 2.8 Cents/Mile

Three different two-VMT pricing schemes, 2-VMT-1 to 2-VMT-3, are considered. As shown in Table 4-20, each scheme is able to reduce the same amount of annual VMT with similar changes in CS, revenue and welfare. The distributional impacts of these two-VMT schemes are presented in Table 4-21 to Table 4-26, which are similar to those of the flat fee. Table 4-27 compares the average changes in CS and demonstrates that, although the two-VMT schemes are still regressive in nature, the degree of regressivity is less than that of a flat fee.

Table 4-20: Total change for two-VMT pricing schemes 1 - 3

VMT fee Scheme	Threshold (mile)	VMT Fee up to Threshold (Cents/mile)	VMT Fee beyond Threshold (Cents/mile)	Total change in CS	Total change in Revenue	Total change in Welfare	% VMT Reduction
Flat Fee	2.8			-2646108	2313856	-332252	10.04
2-VMT-1	10000	2.30	3.50	-2855742	2512368	-343374	10.04
2-VMT-2	12000	2.30	3.75	-2894320	2547032	-347287	10.04
2-VMT-3	15000	2.35	4.10	-2911453	2561617	-349836	10.03

Table 4-21: Average changes by income group (Scheme 1)

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-85.69	-0.67	64.47	-21.22
20,000-39,999	3288	-139.70	-0.46	114.27	-25.43
40,000-59,9999	2468	-210.00	-0.42	180.00	-30.01
60,000-79,999	1800	-278.00	-0.39	247.39	-30.61
80,000 to 200,000	3411	-350.60	-0.27	325.56	-25.10

Table 4-22: Average changes by location (Scheme 1)

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-276.30	248.20	-28.16
Urban	10311	-202.50	176.86	-25.72

Table 4-23: Average changes by income group (Scheme 2)

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-82.64	-0.65	62.33	-20.30
20,000-39,999	3288	-137.10	-0.45	112.26	-24.91
40,000-59,9999	2468	-210.40	-0.42	180.28	-30.13
60,000-79,999	1800	-282.80	-0.40	251.30	-31.55
80,000 to 200,000	3411	-363.40	-0.28	336.73	-26.73

Table 4-24: Average changes by location (Scheme 2)

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-285.10	255.70	-29.45
Urban	10311	-203.90	178.20	-25.76

Table 4-25: Average changes by income group (Scheme 3)

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-81.60	-0.64	61.50	-20.10
20,000-39,999	3288	-134.80	-0.45	110.25	-24.57
40,000-59,9999	2468	-208.20	-0.41	178.33	-29.87
60,000-79,999	1800	-283.80	-0.40	251.86	-31.95
80,000 to 200,000	3411	-372.50	-0.29	344.58	-27.92

Table 4-26: Average changes by location (Scheme 3)

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-292.40	261.79	-30.64
Urban	10311	-203.60	177.98	-25.68

Table 4-27: Average changes by income group

Income Group	HH#	Flat mileage fee of 2.8 cents/mile	2-VMT-1	2-VMT-2	2-VMT-3
0-19,999	2119	-0.84	-0.67	-0.65	-0.64
20,000-39,999	3288	-0.49	-0.46	-0.45	-0.45
40,000-59,9999	2468	-0.40	-0.42	-0.42	-0.41
60,000-79,999	1800	-0.35	-0.39	-0.40	-0.40
80,000 to 200,000	3411	-0.22	-0.27	-0.28	-0.29

4.4.2 Two-VMT Pricing vs. Flat Fee of 4.1 Cents/Mile

Similarly, three different two-VMT pricing schemes (2-VMT-4 to 2-VMT-6) are compared against a flat mileage fee of 4.1 cents/mile. The results are presented in Table 4-28 to Table 4-35. The findings are similar to above. The general conclusion is a well-designed two-VMT pricing scheme can reduce the degree of regressivity experienced by the flat mileage fee.

Table 4-28: Total change for two-VMT pricing schemes 4 - 6

VMT fee Scheme	Threshold (mile)	VMT Fee upto Threshold (Cents/mile)	VMT Fee beyond Threshold (Cents/mile)	Total change in CS	Total change in Revenue	Total change in Welfare	% VMT Reduction
Flat Fee	4.1			-5445869	4437238	-1008631	20.14
2-VMT-4	10000	3.50	5.07	-5719999	4697700	-1022299	20.14
2-VMT-5	12000	3.50	5.44	-5775401	4746358	-1029044	20.15
2-VMT-6	15000	3.55	6.00	-5802717	4770269	-1032448	20.11

Table 4-29: Average changes by income group (Scheme 4)

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-185.30	-1.46	125.78	-59.57
20,000-39,999	3288	-288.60	-0.95	217.10	-71.53
40,000-59,9999	2468	-418.40	-0.83	333.46	-84.96
60,000-79,999	1800	-545.20	-0.77	453.70	-91.54
80,000 to 200,000	3411	-693.00	-0.53	609.12	-83.97

Table 4-30: Average changes by location (Scheme 4)

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-551.70	461.75	-89.72
Urban	10311	-406.20	331.33	-74.93

Table 4-31: Average changes by income group (Scheme 5)

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-182.10	-1.43	123.85	-58.32
20,000-39,999	3288	-285.40	-0.94	214.83	-70.58
40,000-59,9999	2468	-418.50	-0.83	333.62	-84.95
60,000-79,999	1800	-551.40	-0.78	458.45	-92.99
80,000 to 200,000	3411	-711.00	-0.54	624.15	-86.88

Table 4-32: Average changes by location (Scheme 5)

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-563.70	471.49	-92.27
Urban	10311	-408.30	333.43	-74.97

Table 4-33: Average changes by income group (Scheme 6)

Income Group	HH#	Ave. change in CS ($)	Ave. change in CS as % of Avg. Income	Ave. change in Revenue ($)	Ave. change in SW ($)
0-19,999	2119	-180.90	-1.42	122.92	-58.02
20,000-39,999	3288	-282.20	-0.93	212.42	-69.86
40,000-59,9999	2468	-414.90	-0.82	330.82	-84.10
60,000-79,999	1800	-552.30	-0.78	458.94	-93.39
80,000 to 200,000	3411	-724.90	-0.56	635.82	-89.16

Table 4-34: Average changes by location (Scheme 6)

Location	HH#	Ave. change in CS ($)	Ave. change in Revenue ($)	Ave. change in SW ($)
Rural	2775	-574.10	479.80	-94.38
Urban	10311	-408.20	333.51	-74.73

Table 4-35: Average changes by income group

Income Group	HH#	Flat mileage fee of 4.1 cents/mile	2-VMT-1	2-VMT-2	2-VMT-3

0-19,999	2119	-1.66	-1.46	-1.43	-1.42
20,000-39,999	3288	-1.00	-0.95	-0.94	-0.93
40,000-59,9999	2468	-0.81	-0.83	-0.83	-0.82
60,000-79,999	1800	-0.71	-0.77	-0.78	-0.78
80,000 to 200,000	3411	-0.47	-0.53	-0.54	-0.56

5 Other Impacts of Nonlinear Pricing

Using GAMS (Brooke et al., 1992), we implemented algorithms in Appendix A to find tolled UE flow-demand pairs for some nonlinear pricing functions and used the coordinate search (see Section A.5) to find the pricing parameters whose associated UE flow-demand pair maximizes the social benefit. We tested our algorithms using two data sets, one is fictitious and the other is from Hull, Canada.

5.1 Rectangular Network

We used a fictitious network displayed in Figure 5-1 which has 36 OD pairs and a (disconnected) tolling area as shown. The travel time function for each link is of the form $s_a(v_a) = T_a(1 + 0.15(v_a/c_a)^4)$, where the values of T_a and c_a are randomly selected from the intervals (5, 20) and (50, 100), respectively. The demand function for every OD pair is linear, i.e., $D_k(t) = a_k + b_k t$, where a_k and b_k are randomly chosen. For each $k \in K$, we first choose a demand, d_k, randomly from the interval (10, 30) and let τ_k^1 and τ_k^2 denote, respectively, the free-flow and user-equilibrium travel time. The latter assumes that the demand is fixed and equals d_k. Then, a_k and b_k are the intercept and slope of the line that passes through two points, $(\tau_k^1, \mu d_k)$ and (τ_k^2, d_k), where μ is a random number between 2 and 3.

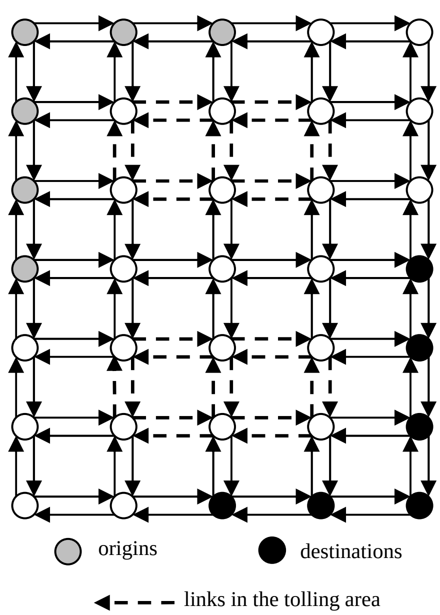


Figure 5-1: Rectangular network with tolling area

For the above rectangular network, the social surplus without any pricing intervention is 19,964.15 units while the maximum obtainable social surplus (i.e., the surplus associated with system optimum) is 43,421.54. For locally optimal linear and nonlinear pricing, the results are listed below:

- Under a (locally) optimal linear pricing (i.e., $T(l) = 1.0\ell$), the social the social surplus is 26,545.47 units. When compared to the case without any tolling intervention, linear pricing increases the social surplus from 19,964.15 to 26,545.47 or by approximately 33%.

- Under a (locally) optimal two-part pricing scheme (i.e., $T(\ell) = 32 + 0.2\ell$), the social surplus is 27,544.09 units. When compared to the case without any tolling intervention, two-part pricing increases the social surplus from 19,964.15 to 27,544.09 or by approximately 38%.

- Under a (locally) optimal two-VMT pricing scheme (i.e., $T(\ell) = \max\{1.05\ell, -2.5 + 1.1\ell\}$), the social surplus is 26,608.74 units. When compared to the case without any tolling intervention, two-VMT pricing increases the social surplus from 19,964.15 to 26,608.74 or by approximately 33%.

For our rectangular network, two-part pricing offers the best social surplus while linear and two-VMT pricing yield similar social surpluses.

5.2 Hull Network

A road network well known in the literature is the planning network from Hull, Canada. The network contains 501 nodes (of which 23 are centroids) and 789 links. There are 142 OD pairs with positive travel demands. The tolling area consists of Eddy, Frontenac, Gagnon, Hotel de Ville, Laval, LeDuc, Papineau, Pilon, Victoria, Wellington and Wright depicted in Figure 5-2.

For Hull, the social surplus without any pricing intervention is 6,935.13 units while the maximum obtainable social surplus is 17,902.45. For the optimal linear and nonlinear pricing, the results are listed below:

- Under a (locally) optimal linear pricing (i.e., $T(l) = 5.0\ell$), the social the social surplus is 7,503.06 units. When compared to the case without any tolling intervention, linear pricing increases the social surplus from 6,935.13 to 7,503.06 or by approximately 8%.

- Under a (locally) optimal two-part pricing scheme (i.e., $T(\ell) = 0.5 + 0.5\ell$), the social surplus is 7,770.17 units. When compared to the case without any tolling intervention, two-part pricing increases the social surplus from 6,935.13 to 7,770.17 or by approximately 12%.

- Under a (locally) optimal two-VMT pricing scheme (i.e., $T(\ell) = \max\{0.25, -1.0 + 3.1\ell\}$), the social surplus is 7,021.89 units. When compared to the case without any tolling intervention, two-VMT pricing increases the social surplus from 6,935.13 to 7,021.89 or by approximately 1.2%.

Similar to the rectangular network, two-part pricing offers the best social surplus. However, the two-VMT pricing yield a social surplus significantly worse than that of linear pricing. In general, the performance of various pricing schemes depends on the network topology as well as other parameters. Based on the above two networks, two-part pricing seems to offer a higher social surplus.

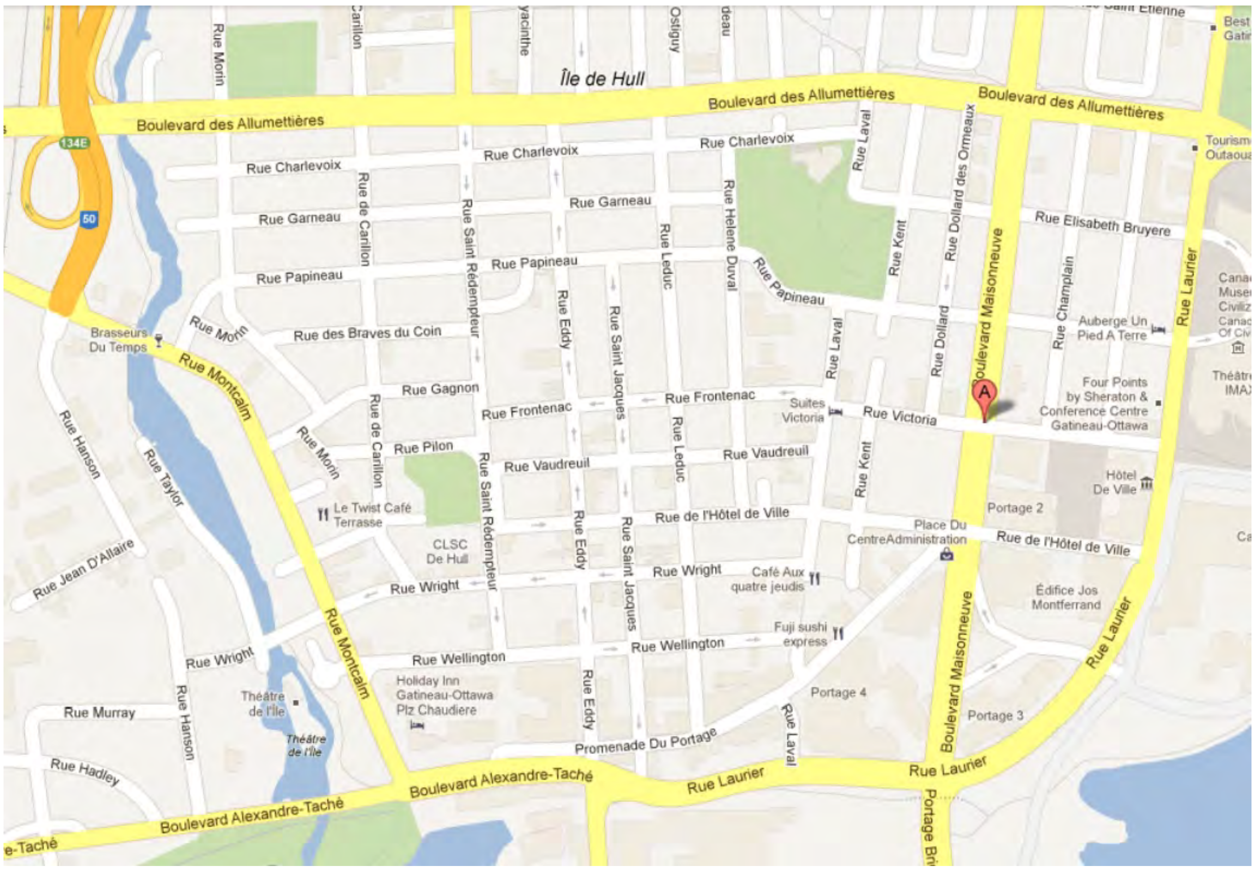


Figure 5-2: Hull network

6 Conclusions

In general, finding tolled user equilibrium under nonlinear pricing is a difficult problem because nonlinear tolling functions are not link-wise additive. In the literature, several (e.g., Gabriel and Bernstein, 1997a, 1997b, Lo and Chen, 2000 and Agdeppa et al., 2007) proposed algorithms to find tolled user equilibrium solutions for general nonlinear tolling functions. These algorithms are in general complicated (e.g., using techniques not covered in graduate courses on transportation planning and analysis), time-consuming and require assumptions that may not hold in practice.

By focusing on simple nonlinear function such as those in Section 3, our research shows that

- Finding tolled user equilibrium solutions under two-part pricing (Figure 3-2) is relatively easy to solve. Existing algorithms such as those used in software packages can be modified to solve the problem.
- For, e.g., nonlinear pricing in Figure 3-3 and Figure 3-4, finding tolled user equilibrium is more difficult and requires more computational time. On the other hand, our algorithms for solving these problems are simpler than those in the literature.
- Using simple nonlinear tolling functions also allows us to develop an algorithm to find the best function to, e.g., maximize the social surplus, minimize traffic pollution, etc.
- Data from Florida in 2009 indicate that two-VMT pricing, when properly designed, can be less regressive than flat mileage fee.

7 References

Agdeppa, R.P., Yamashita, N., Fukushima, M. (2007) “The traffic equilibrium problem with nonadditive costs and its monotone mixed complementarity problem formulation.” *Transportation Research Part B*, 41, 862 – 874.

Ahuja, R. K., Magnanti, T.L, Orlin, J.B. (1993) *Network Flows: Theory, Algorithms, and Applications*. New Jersey: Prentice Hall.

Bazaraa, M. S.; Jarvis, J.S., Sherali, H.D. (2010) *Linear Programming and Network Flows*. New York: John Wiley & Sons.

Bazaraa, M. S.; Sherali, H.D, Shetty, C.M. (2006) *Nonlinear Programming: Theory and Algorithms*. New York: John Wiley & Sons.

Brooke, A., Kendrick, D. and Meeraus, A. (1992). *GAMS: A User’s Guide*. South San Francisco: The Scientific Press

De Borger, B., (2001) “Discrete choice models and optimal two-part tariffs in the presence of externalities: Optimal taxation of cars.” Regional Science and Urban Economics 31, 471–504.

Drud, A. (1992) “A large-scale GRG code.” *ORSA Journal on Computing*, 6, 207 - 216.

Dupuit, J., (1894) “On tolls and transport charges,” Annales des Pont et Chaussees, 17. (Also translated in International Economic Papers, Macmillian, London, 1952)

Facchinei, F., Pang, J.-S. (2003). *Finite-Dimensional Variational Inequalities and Complementarity Problems*,Volumes I and II. New York: Springer-Verlag.

Florian, M., Hearn, D.W. (2003) “Network equilbrium and pricing.” In R. W. Hall (Ed.), *Handbook of Transportation Science* (pp. 373 – 411), New York: Springer.

Frank, M., Wolfe, P. (1956) “An algorithm for quadratic programming.” *Naval Research Logistics Quarterly*, 3, 95-110.

Gabriel, S.A., Bernstein, D., (1997a) “The traffic equilibrium problem with nonadditive path costs.” *Transportation Science*, 31(4), 337 – 348.

Gabriel, S.A., Bernstein, D., (1997b) “Solving the nonadditive traffic equilibrium problem.” In P.M. Pardalos, D.W. Hearn, W.W. Hager (eds.), *Network Optimization*, Lecture Notes in Economics and Mathematical Systems, 450, (pp. 72 – 102), New York: Springer-Verlag.

Hearn, D. W., Lawphongpanich, S., Ventura, S. (1987) “Restricted simiplicial decomposition: computation and extensions.” *Mathematical Programming Study*, 31, 99 - 118.

IBM Corp. (2009) “User’s manual for CPLEX.” Armonk, New York.

Lawphongpanich, S. , Hearn, D.W. (1984) “Simplicial decomposition of the asymmetric traffic assignment problem.” *Transportation Research Part B*, 18, 123-133.

Lo, H.K., Chen, A. (2000) “Traffic equilibrium problem with route-specific costs: formulation and algorithms.” *Transportation Research Part B*, 34, 493 – 513.

Luo, Z.-Q., Pang, J.-S., Ralph, R. (1996) *Mathematical Programs with Equilibrium Constraints*. Cambridge University Press, Cambridge

Maruyama, T., Harata, N. (2006) “Difference between area-based and cordon-based congestion pricing: Investigation by trip-chain-based network equilibrium model with non-additive path costs.” *Transportation Research Record*, 1964, 1 - 8.

Maruyama, T., Sumalee, A. (2007) “Efficiency and equity comparison of cordon- and area-based road pricing schemes using a trip-chain equilibrium model.” *Transportation Research Part A*, 41, 655 - 671.

Menon, , A. P.G., Lam, S.H., Fan, H.S.L. (1993) “Singapore's road pricing system: its past,

present and future." *Institute of Traffic Engineer Journal*, 63, 44 - 48.
Patriksson, M.(1994) The Traffic Assignment Problem: Models and Methods. Utrecht: VSP.
Pigou, A., (1920) The Economics of Welfare, Macmillian, London.
Rinnooy Kan, A. H.G., Timmer, G.T. (1989) "Global optimization." In G. L. Nemhauser, A. H.G. Rinnooy Kan and M. J. Todd (Eds.), *Optimization*, (pp. 631 – 662), New York: Elsevier Science Publishers.
Santos, G., Shaffer, B. (2004) "Preliminary results of the London congestion charging scheme." *Public Works Management & Policy*, 9, 164 - 181.
Stockholmsforsoket. (2006) www.stockholmsforsoket.se/templates/page.aspx?id=10215.
Von Hohenbalken, B. (1977) "Simplicial decomposition in nonlinear programming algorithms." *Mathematical Programming*, 13, 49 – 68.
Wang, J., Lindsey, J., Yang, H., (2011) "Nonlinear pricing on private roads with congestion and toll collection costs," *Transportation Research Part B*, 45, 4 - 90.
Wilson, R., (1993) *Nonlinear Pricing*, Oxford University Press, New York.
Zhang, X., Yang, H. (2004) "The optimal cordon-based network congestion pricing problem." *Transportation Research Part B*, 38, 517 - 537.

Appendix A: Technical Development

For this appendix, Section A.1 describes the pricing functions considered in this report. Section A.2 defines our notation and states path-based UE conditions for later reference. Section A.3 formulates the UE problem in terms of path flows and modifies simplicial decomposition to find a UE flow-demand pair under our nonlinear pricing functions. Section A.4 states link-based UE conditions and discusses when these conditions are equivalent to those based on paths. Section A.5 presents a search algorithm for finding optimal pricing parameters, e.g., those that maximize the social benefit. To illustrate the simplicity of using link-based conditions and problems, Section A.6 gives a version of the Frank-Wolfe algorithm (a well-known algorithm for linearly constrained convex programs) for solving the UE problem with two-part pricing.

A.1 Nonlinear Pricing Functions

The tolling function, $T(\ell)$, in this report is of the form:

$$T(\ell) = \begin{cases} T^{min}(\ell) \text{ or } T^{max}(\ell), & \ell > 0 \\ 0, & \ell \leq 0 \end{cases} \tag{A.1.1}$$

where $T^{min}(\ell) = \min\{\beta_1 + \mu_1\ell, \beta_2 + \mu_2\ell\}$ and $T^{\max}(\ell) = \max\{\beta_1 + \mu_1\ell, \beta_2 + \mu_2\ell\}$. Recall that ℓ is the distance traveled inside the tolling area. (Herein, distances are measured in miles and we refer to a rate or fee based on miles traveled as a "VMT fee", where VMT is an abbreviation for "vehicle-mile traveled.") In both $T^{min}(\ell)$ and $T^{max}(\ell)$, μ_1 and μ_2 are nonnegative VMT fees. Typically, β_1 and β_2 are nonnegative. However, one may be negative to reproduce some tolling functions in practice more accurately. (See the discussion about three-part tariffs below.)

Both $T^{min}(\ell)$ and $T^{max}(\ell)$ are piecewise linear functions with two linear pieces. Although the number of linear pieces can be larger, i.e., $T^{min}(\ell) = \min\{\beta_1 + \mu_1\ell, \cdots, \beta_n + \mu_n\ell\}$ and $T^{max}(\ell) = \max\{\beta_1 + \mu_1\ell, \cdots, \beta_n + \mu_n\ell\}$, where $n \geq 2$, we set $n = 2$ in this report for two reasons. First, the results for $n = 2$ can be extended to the cases with larger n without much difficulty. As cautioned in Wilson (1993), the second reason is that large n is often *not* practical. Pricing functions with many linear pieces generally result in tolling schemes too complex for motorists to understand and respond properly. Moreover, pricing functions with only a few linear pieces can typically capture most of the benefits offered by those with many.

When β_1, μ_1, β_2 and μ_2 are chosen appropriately, $T^{\min}(\ell)$ and $T^{\max}(\ell)$ capture common nonlinear pricing functions in the economics and road pricing literature (see, e.g., Wilson, 1993 and Wang et al., 2011). Figure A-1 displays tolling functions based on $T^{min}(\ell)$.

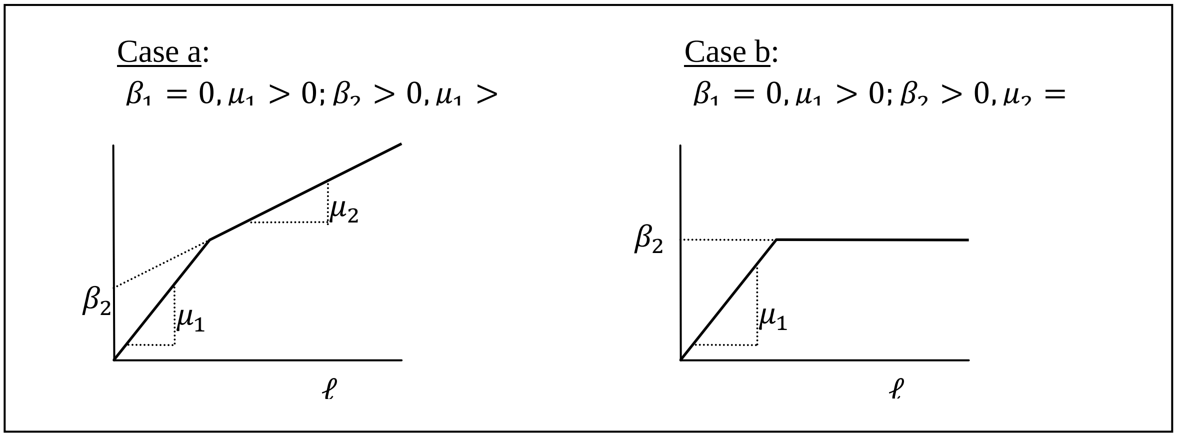


Figure A-1: Pricing functions based on $T^{\min}(\ell)$

In case (a), the VMT fee for a longer distance (μ_2) is smaller than the one for a shorter distance (μ_1), i.e., heavy road users receive discounts. Case (b) allows users to either pay a VMT fee at a rate μ_1 or a fixed fee, β_2, for unlimited travel inside the tolling area. The former is more economical when the travel distance is sufficiently short, i.e., less than the point where $\mu_1 \ell = \beta_2$. Although both cases may be suitable for many industries, it is not clear that they would be adopted for congestion mitigation.

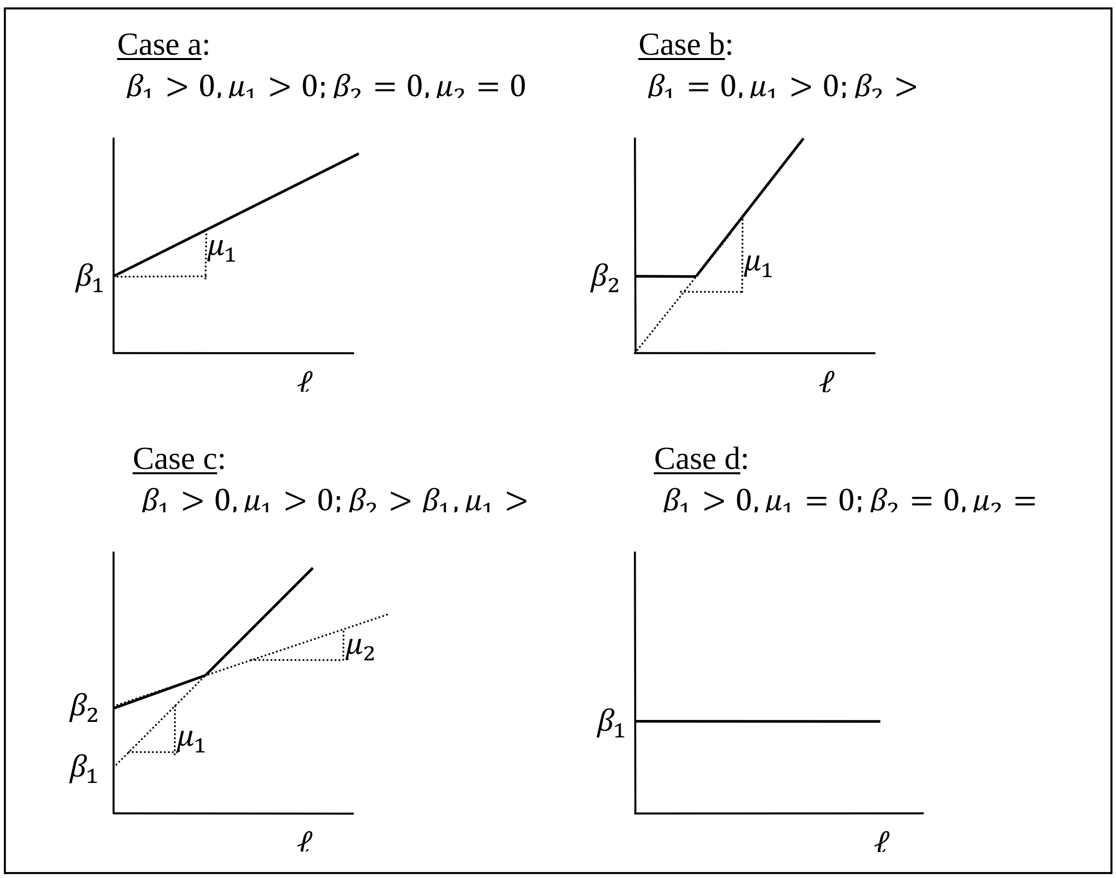


Figure A-2: Pricing functions based on $\boldsymbol{T}^{\mathbf{max}}(\boldsymbol{\ell})$

For the pricing functions based on $T^{\max}(\ell)$ in Figure A-2, case (a) requires users to pay two fees. One is an access fee (β_1) and the other is a VMT fee (μ_1). Economists commonly refer to this form of pricing as a two-part tariff or pricing scheme. Similarly, the function in case (b) also consists of an access and VMT fee. However, the latter only applies when the travel distance exceeds a threshold, a point where $\beta_1 + \mu_1\ell = \beta_2$. (When β_2 and μ_1 are fixed, β_1 may need to be negative to achieve a desired threshold value.) In economics, some refer to case (b) as a three-part tariff. Instead of giving discounts to heavy users, case (c) discourages heavy road usage by charging a higher VMT fee (μ_1) when the travel distance exceeds a threshold, a point where $\beta_1 + \mu_1\ell = \beta_2 + \mu_2\ell$. Finally, the pricing function for case (d) is suitable for area-based pricing (see, e.g., Maruyama and Sumalee, 2007), a tolling scheme under which users can enter and use the tolling area as often and as much as they like during a specified period after paying an access fee, β_1. (Area-based pricing is different from cordon pricing. For the latter, users generally pay a fee each time they enter the tolling area.) In addition to those shown in the two figures, setting β_1, β_2, and μ_2 to zero reduces $T(\ell)$ to linear pricing, i.e., $T(\ell) = \mu_1\ell$.

A.2 Path-Based User Equilibrium Conditions Under Nonlinear Pricing

This Section states UE conditions under nonlinear pricing using path flows. Doing so allows us to define our notation and provide information for discussion in subsequent sections.

Let Ω be the set of links (or arcs) in the road network. A link in Ω is denoted as a or a pair (i,j), where i and j are nodes corresponding to the start and end of a road segment. For travel demands, K denotes the set of origin-destination (OD) pairs and d_k is the demand for OD pair $k \in K$. Associated with each OD pair, there is an inverse demand function $D_k^{-1}(\cdot)$. Additionally, $\boldsymbol{d} \in R_+^{|K|}$ and $\boldsymbol{D}^{-1}(\cdot) \in R_+^{|K|}$ are vectors of these demands and their inverse functions, respectively. (Herein, the bold typeface indicates vectors of variables or functions and the plus sign in the subscript indicates that each component of the vector is nonnegative.)

To satisfy demands, P^k denotes the set of all possible paths for OD pair k. Then, f_r^k represents the number of travelers using path $r \in P^k$ and $\boldsymbol{f}$ is a vector of these path flows. Then, the set of all feasible flow-demand pairs, $(\boldsymbol{f}, \boldsymbol{d})$, can be described as follows:

$$V^f = \left\{(\boldsymbol{f}, \boldsymbol{d}): \sum_{r \in P^k} f_r^k = d_k, f_r^k \geq 0, d_k \geq 0, \forall k \in K, r \in P^k\right\}. \tag{A.2.1}$$

In words, $(\boldsymbol{f}, \boldsymbol{d})$ is a feasible flow-demand pair if the sum of the flows on all paths connecting the origin of OD pair k to its destination equals d_k and both $\boldsymbol{f}$ and $\boldsymbol{d}$ are nonnegative. It is also convenient to refer to a flow-demand pair as $(\boldsymbol{v}, \boldsymbol{d})$, where $\boldsymbol{v}$ a vector of the aggregate link flows, v_a. By letting $\delta_{ar} = 1$ if arc a is on path r and $\delta_{ar} = 0$ otherwise, it is possible to describe V^f as follows:

$$V^f = \left\{(\boldsymbol{v}, \boldsymbol{d}): v_a = \sum_k \sum_{r \in P^k} \delta_{ar} f_r^k, \sum_{r \in P^k} f_r^k = d_k, f_r^k \geq 0, d_k \geq 0, \forall k \in K, r \in P^k\right\}. \quad \text{(A.2.2)}$$

We use both definitions of V^f interchangeably throughout this report and refer the elements of V^f either as $(\boldsymbol{v}, \boldsymbol{d})$ or $(\boldsymbol{f}, \boldsymbol{d})$.

Associated with each arc, there is a travel time or link performance function, $s_a(\cdot)$, and $\boldsymbol{s}(\cdot) \in R_{++}^L$ is a vector of these functions, where L is the cardinality of Ω and the "++" sign in the subscript indicates that each component of the vector is positive. In addition, l_a denotes the length of arc a and $l_a > 0$ for all $a \in \Omega$. For tolling, Ω is partitioned into two subsets, Ω^1 and Ω^2, where the former contains links inside the tolling area and the later consists of those outside. By definition, $\Omega^1 \cap \Omega^2 = \emptyset$ and $\Omega = \Omega^1 \cup \Omega^2$. As mentioned previously, arcs in Ω_1 need not be connected. Similarly, P^k is divided into two subsets: TP^k and NP^k. The former, TP^k, consists of paths containing arcs in Ω^1 and using these paths requires paying tolls. In general, paths in TP^k contain links in both Ω^1 and Ω^2 to connect the origin of OD pair k to its destination. On the other hand, paths in NP^k contain no link in Ω^1 and are thus toll-free. Given a pricing function $T(\cdot)$, $(\boldsymbol{v}, \boldsymbol{d}) \in V^f$ is in tolled UE if the following conditions hold:

$$T\left(\sum_{a \in \Omega^1} \delta_{ar} l_a\right) + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(d_k) \quad \forall r \in TP_{++}^k(\boldsymbol{v}, \boldsymbol{d}), k \in K, \quad \text{(A.2.3)}$$

$$T\left(\sum_{a \in \Omega^1} \delta_{ar} l_a\right) + \sum_{a \in \Omega} \delta_{ar} s_a(\boldsymbol{v}) \geq D_k^{-1}(d_k) \quad \forall r \in TP_0^k(\boldsymbol{v}, \boldsymbol{d}), k \in K, \quad \text{(A.2.4)}$$

$$\sum_{a \in \Omega^2} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(d_k) \quad \forall r \in NP_{++}^k(\boldsymbol{v}, \boldsymbol{d}), k \in K, \quad \text{(A.2.5)}$$

$$\sum_{a \in \Omega^2} \delta_{ar} s_a(\boldsymbol{v}) \geq D_k^{-1}(d_k) \quad \forall r \in NP_0^k(\boldsymbol{v}, \boldsymbol{d}), k \in K. \quad \text{(A.2.6)}$$

In (A.2.3), $TP_{++}^k(\boldsymbol{v}, \boldsymbol{d})$ denotes the set of utilized toll paths with respect to $(\boldsymbol{v}, \boldsymbol{d}) \in V^f$, i.e., $TP_{++}^k(\boldsymbol{v}, \boldsymbol{d}) = \{r \in TP^k: f_r^k > 0, r \in P^k\}$. Similarly, $TP_0^k(\boldsymbol{v}, \boldsymbol{d})$ in (A.2.4) is the set of paths not utilized and $TP_0^K(\boldsymbol{v}, \boldsymbol{d}) = \{r \in TP^k: f_r^k = 0, r \in P^k\}$. In (A.2.5) and (A.2.6), $NP_{++}^k(\boldsymbol{v}, \boldsymbol{d})$ and $NP_0^k(\boldsymbol{v}, \boldsymbol{d})$ are similarly defined for toll-free paths. The expression on the left hand side of (A.2.3) and (A.2.4) consists of the toll amount and travel time for path $r \in TP^k$. (In this report, tolls are measured in units of time.) Because paths in NP^k are toll free, their costs or the summations on the left hand side of (A.2.5) and (A.2.6) consist solely of travel times. In words, (A.2.3) and (A.2.5) state that, at equilibrium, all utilized paths (toll or not) must have the same generalized cost that equals to the value of the inverse demand function evaluated at the "realized" demand d_k. Conditions (A.2.4) and (A.2.6) imply that the costs of those not utilized cannot be lower than $D_k^{-1}(d_k)$.

When $T(\cdot)$ is nonlinear, the generalized cost expressions on the left hand side of conditions (A.2.3) and (A.2.4) are not link-wise additive and it may be intuitive to conclude that tolled UE conditions based on link flows do not exist (see, e.g., Maruyama and Sumalee, 2007). However, results in Section 8.4 show otherwise.

To simplify our presentation and highlight key ideas, assume that $s_a(\cdot)$ is a function only of v_a, i.e., the Jacobian of $\boldsymbol{s}(\boldsymbol{v})$ is diagonal. Under this assumption, finding a toll-free equilibrium flow-demand pair reduces to a convex optimization problem. An extension to, e.g., the case with an asymmetric and positive definite Jacobian is straightforward and generally involves finding solutions to variational inequalities or VIs (see, e.g., Florian and Hearn, 2003, and Patriksson, 1994). In addition, $D_k^{-1}(\cdot)$ is assumed to be nonincreasing and $0 \leq D_k^{-1}(d) < \infty, \forall d \geq 0$.

Henceforth, we assume that $T(\ell)$ is based on $T^{max}(\ell)$, a function more suitable for managing travel demand, reducing congestion, and lessening the environmental impacts. Although most discussion and many results herein extend in an obvious manner to the case with $T^{min}(\ell)$, the resulting optimization problems and VIs generally minimize a nonconvex objective function and are defined with functions whose Jacobians are indefinite, respectively. Solving such optimization problems with, e.g., commercial software may not yield globally optimal solutions and VIs with indefinite Jacobians are not well solved (see, e.g., Facchinei and Pang, 2003).

A.3 Finding Equilibrium Flow-Demand Pairs Using Path Flows

This section assumes that $T(\ell)$ is based on $T^{max}(\ell)$ defined in Section 8.2 and modifies traditional algorithms such as simplicial decomposition (see, e.g., von Hohenbalken, 1977, Lawphongpanich and Hearn, 1984, Hearn et al., 1987, and Patriksson, 1994) to find a UE flow-demand pair. This is advantageous for two reasons. The underlying concepts in traditional algorithms are well understood and, as demonstrated below, they work well when $T(\ell)$ is defined with $T^{max}(\ell)$. The former also makes the software development easier because existing computer programs for traditional algorithms can be modified to include nonlinear pricing.

For each path $r \in P^k$, its travel distance inside the tolling area, $\sum_{a\in\Omega^1} \delta_{ar} l_a$, is fixed. Thus, the toll, τ_r, for path r is also fixed. Specifically, $\tau_r = 0, \forall r \in NP^k$, and $\tau_r = T(\sum_{a\in\Omega^1} \delta_{ar} l_a)$ is nonnegative for all $r \in TP^k$. Then, the tolled user equilibrium (TUE) problem, i.e., the problem of finding a UE flow-demand pair with a given pricing function $T(\cdot)$, can be formulated in terms of path flows as follows:

$$
\begin{aligned}
TUE: \quad min \quad & \sum_{a\in\Omega}\int_0^{v_a} s_a(\omega)d\omega - \sum_{k\in K}\int_0^{d_k} D_k^{-1}(\omega)d\omega + \sum_{k\in K}\sum_{r\in P^k}\tau_r f_r^k \\
s.t. \quad & \sum_{r\in P^k} f_r^k - d_k = 0, \forall k \in K \\
& v_a = \sum_k \sum_{r\in P^k} \delta_{ar} f_r^k, \forall a \in \Omega \\
& f_r^k \geq 0, \forall k \in K, r \in P^k
\end{aligned} \tag{A.3.1}
$$

Without the last term in the objective function, the above problem reduces to a problem for finding a (toll-free) UE flow-demand pair when demands are elastic (see, e.g., Florian and Hearn, 2003). Under the assumptions stated at the end of Section 8.3, the functions in the first and second summations in the objective are convex. The last summation calculates the toll revenue and is linear with respect to f_r^k, the path-flow variables. The two main sets of constraints ensure feasibility and convert path flows into aggregate link flows. Moreover, the Karush-Kuhn-Tucker or KKT conditions (see, e.g., Bazaraa et al., 2006) are both necessary and sufficient for TUE because it is a linearly constrained convex program. Using the fact that $\tau_r = T(\sum_{a\in\Omega^1}\delta_{ar}l_a)$ and $P^k = TP^k \cup NP^k, \forall k \in K$, it is relatively simple to demonstrate that the KKT conditions for TUE reduce to (A.2.3) – (A.2.6). Thus, an optimal solution to TUE is a UE flow-demand pair under the pricing function $T(\cdot)$.

Below is a version of simplicial decomposition (SD) that generates the necessary paths between every OD pair. (For other variations, see, e.g., Lawphongpanich and Hearn, 1984, and Hearn et al., 1987.) Briefly, the algorithm starts with a zero flow-demand pair in Step 1 (i.e., there is no travel demand initially) and solves an optimization problem to generate new paths in Step 2 for all OD pairs. In Step 3, the algorithm stops when paths generated in Step 2 cannot further reduce the objective value of TUE. If the algorithm does not stop, Step 4 adds paths from Step 2 to Π^k, the set of indices associated with the generated paths for OD pair k. Typically, $\Pi^k \subset P^k, \forall k \in K$. In Step 5, the algorithm solves an approximate version of TUE in which P^k is replaced by Π^k and returns to Step 2 where the process repeats.

Simplicial Decomposition for TUE

Step 1: Set $(\boldsymbol{v}^1, \boldsymbol{d}^1) = (0, 0)$ and $n = 1$. For each OD pair k, set $\Pi^k = \emptyset$ and $r^k = 1$.

Step 2: For each OD pair k, let $(\boldsymbol{z}^k, w^k)$ solve the following (sub)problem and c^k denotes its optimal objective value:

$$\begin{aligned} c^k = \quad & min \quad \sum_{a \in \Omega} s_a(\boldsymbol{v}^n) z_a^k + w^k \\ & s.t. \quad A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & \qquad \sum_{a \in \Omega^1} z_a^k \leq M q^k \\ & \qquad \beta_1 q^k + \mu_1 \sum\nolimits_{a \in \Omega^1} l_a z_a^k \leq w^k \\ & \qquad \beta_2 q^k + \mu_2 \sum\nolimits_{a \in \Omega^1} l_a z_a^k \leq w^k \\ & \qquad q^k \in \{0,1\}, z_a^k \in \{0,1\}, \forall a \in \Omega \end{aligned} \tag{A.3.2}$$

Step 3: If $c^k - D_k^{-1}(d_k^n) \geq 0, \forall k \in K$, stop and the current solution $(\boldsymbol{v}^n, \boldsymbol{d}^n)$ is a tolled UE flow-demand pair. Otherwise, go to Step 4.

Step 4: For each OD pair k such that $c^k - D_k^{-1}(d_k^n) < 0$, set $\delta_{ar^k} = z_a^k,\ \tau_{r^k} = w^k,\ \Pi^k = \Pi^k \cup \{r^k\}$, and $r^k = r^k + 1$.

Step 5: Let $(\boldsymbol{v}^{n+1}, \boldsymbol{d}^{n+1})$ solve the (master) problem below, set $n = n + 1$, and return to Step 2.

$$\begin{aligned} min \quad & \sum_{a \in \Omega} \int_0^{v_a} s_a(\omega) d\omega - \sum_{k \in K} \int_0^{d_k} D_k^{-1}(\omega) d\omega + \sum_{k \in K} \sum_{r \in \Pi^k} \tau_r f_r^k \\ s.t. \quad & \sum_{r \in \Pi^k} f_r^k - d_k = 0, \forall k \in K \\ & v_a = \sum_k \sum_{r \in \Pi^k} \delta_{ar} f_r^k, \forall a \in \Omega \\ & f_r^k \geq 0, \forall k \in K, r \in \Pi^k \end{aligned} \tag{A.3.3}$$

In the above, Step 1 uses a zero flow-demand pair to initialize the algorithm. Subsequently, the link travel times, $s_a(\boldsymbol{v}^n)$, in the subproblem in Step 2 (or, more descriptively, the path-generation problem) are free-flow travel time during the first iteration, i.e., when $n = 1$. For each OD pair, the subproblem finds a path with the least generalized cost. The first summation in the objective function computes the path travel time and w^k is the toll amount. In the first constraint, A is the node-arc incidence matrix of the road network and $\boldsymbol{E}_k \in R^N$ is an (input-output) vector with exactly two nonzero components. The component corresponding to the origin node of the OD pair k contains a "1" and the one for the destination contains a "-1." Thus, the first constraint balances the flows into and out of each node. The binary variable q^k in the second constraint indicates whether to pay tolls and M is a sufficiently large positive constant, e.g., $M = |\Omega^1| + 1$. Setting $q^k = 0$ forces z_a^k to be zero for all $a \in \Omega^1$, i.e., the path does not enter the tolling area. With $q^k = 0$ and $z_a^k = 0, \forall a \in \Omega^1$, the left-hand sides of the next two constraints (the 3rd and 4th constraints) reduce to zero. Consequently, w^k must be zero to minimize the objective function and the path associated with $\boldsymbol{z}^k$ is toll-free. When $q^k = 1$, z_a^k for $a \in \Omega^1$ are allowed to be one, i.e., the path can use links in the tolling area, and the combination of the 3rd and 4th constraints ensure that

$$\max\left\{\beta_1 + \mu_1 \sum\nolimits_{a\in\Omega^1} l_a z_a^k, \beta_2 + \mu_2 \sum\nolimits_{a\in\Omega^1} l_a z_a^k\right\} \le w^k. \quad \text{(A.3.4)}$$

As before, the inequality "≤" in the above expression must hold at equality to minimize the objective function, i.e., w^k is the toll amount associated with $\boldsymbol{z}^k$.

The stopping criterion in Step 3 ensures that all paths, i.e., those in the current set Π^k and otherwise, cost no less than $D_k^{-1}(d_k^n)$. This implies that no path can lead to a smaller objective value. Then, the fact that $(\boldsymbol{v}^n, \boldsymbol{d}^n)$ solves the master problem ensures, via its KKT conditions, that the solution satisfies the tolled UE conditions. Also, it is more practical to replace the stopping criterion in Step 3 with $c^k - D^{-1}(d_k^n) \ge -\epsilon$, where ϵ is a sufficiently small positive constant, e.g., $\epsilon = 10^{-6}$.

Step 4 adds an additional path to the set Π^k and performs the necessary updates. Finally, the master problem in Step 5 is a convex optimization problem with linear constraints, a class of problems relatively easy to solve. As mentioned previously, the master problem is also an approximation of the TUE problem.

The above SD algorithm converges to an optimal solution in a finite number of iterations. The argument is similar to those in the literature (see, e.g., Lawphongpanich and Hearn, 1984) and follows from three facts. First, the number of paths without cycles is finite. (Recall that we assume that the link performance function $s_a(\cdot)$ is positive for all $a \in \Omega$. Thus, the solutions to the problem in Step 2 must correspond to paths without cycles.) Second, because SD never eliminates paths from Π^k, new paths generated in Step 2 must be distinct from those in the current Π^k. Finally, the optimal objective value of the master problem strictly decreases at the end of every iteration prior to termination because newly added paths in Step 4 satisfy $c^k - D_k^{-1}(d_k^n) < 0$, i.e., a condition that ensures a decrease in the objective value.

A.3.1 Solving Path Generating Problems

Consider the path-generating problem (PG) in Step 2. Although it is possible to solve PG as a single problem, our numerical experiments indicate that it is more efficient to obtain an optimal solution to PG by solving two smaller problems for each OD pair, one contains binary variables and the other does not. Solving these two problems is akin to solving PG twice, once using the tolling area ($q^k = 1$) and another not using it ($q^k = 0$). Then, the better of the two optimal solutions is the solution to PG.

When $q^k = 1$, the third constraint in PG becomes $\sum_{a\in\Omega^1} z_a^k \le M$. When M is sufficiently large, the constraint is never binding and can be eliminated. Consequently, PG reduces to the following:

$$\begin{aligned}
SUB1(\boldsymbol{v}^n): \quad min \quad & \sum\nolimits_{a\in\Omega} s_a(\boldsymbol{v}^n) z_a^k + w^k \\
s.t. \quad & A\boldsymbol{z}^k = \boldsymbol{E}_k \\
& \beta_1 + \mu_1 \sum\nolimits_{a\in\Omega^1} l_a z_a^k \le w^k \\
& \beta_2 + \mu_2 \sum\nolimits_{a\in\Omega^1} l_a z_a^k \le w^k \\
& z_a^k \in \{0,1\}, \forall a \in \Omega
\end{aligned} \tag{A.3.5}$$

The above problem can be viewed as a generalization of a shortest path problem with two side constraints (see, e.g., Ahuja et al., 1993), a NP-complete problem. When compared to other NP-complete problems, our numerical experiments indicate that commercial software such as CPLEX (IBM, 2009) can solve $SUB1(\boldsymbol{v}^n)$ efficiently because the second and third constraints can be satisfied easily. For any binary $\boldsymbol{z}^k$ feasible to the first constraint, setting $w^k = \max\{\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k, \beta_2 + \mu_2 \sum_{a\in\Omega^1} l_a z_a^k\}$ yields a pair $(\boldsymbol{z}^k, w^k)$ feasible to $SUB1(\boldsymbol{v}^n)$.

For the other case ($q^k = 0$), we partition A into two submatrices, A_1 and A_2, where A_n is the node-arc incidence matrix for the network induced by arcs in Ω^n, where $n = 1, 2$. Thus, A can be written as $[A_1 \colon A_2]$. Similarly, we also partition $\boldsymbol{z}^k$ as follows:

$$\boldsymbol{z}^k = \begin{bmatrix} \boldsymbol{z}_1^k \\ \boldsymbol{z}_2^k \end{bmatrix}. \tag{A.3.6}$$

In the above, $\boldsymbol{z}_1^k$ is a (sub)vector consisting of variables z_a^k for $a \in \Omega^1$. The similar holds for $\boldsymbol{z}_2^k$. Under this partitioning, the flow-balance constraint becomes $A_1\boldsymbol{z}_1^k + A_2\boldsymbol{z}_2^k = \boldsymbol{E}_k$. When $q^k = 0$, the path cannot enter the tolling area. Thus, $\boldsymbol{z}_1^k = 0$ and the subproblem in Step 2 reduces to the following because the constraints involving arcs in Ω^1 are irrelevant and thus eliminated:

$$\begin{aligned}
SUB2(\boldsymbol{v}^n): \quad min \quad & \sum_{a\in\Omega^2} s_a(\boldsymbol{v}^n) z_a^k \\
s.t. \quad & A_2\boldsymbol{z}_2^k = \boldsymbol{E}_k \\
& z_a^k \in \{0,1\}, \forall a \in \Omega^2
\end{aligned} \tag{A.3.7}$$

Note that A_2 is totally unimodular because it is a submatrix of A, a totally unimodular matrix. Thus, basic solutions to $A_2\boldsymbol{z}_2^k = \boldsymbol{E}_k$ are always integral and the binary restriction for z_a^k is unnecessary. In other words, $SUB2(\boldsymbol{v}^n)$ can be equivalently written as follows:

$$\begin{aligned} SUB2a(\boldsymbol{v}^n): \quad & min \quad \sum_{a \in \Omega^2} s_a(\boldsymbol{v}^n) z_a^k \\ s.t. \quad & A_2 \boldsymbol{z}_2^k = \boldsymbol{E}_k \\ & z_a^k \geq 0, \forall a \in \Omega^2 \end{aligned} \tag{A.3.8}$$

Observe that a unit upper bound on z_a^k is unnecessary in $SUB2a(\boldsymbol{v}^n)$ because $\boldsymbol{E}_k$ implies that there is only one unit of flow in the problem. Instead of solving PG directly, we solve $SUB1(\boldsymbol{v}^n)$ and $SUB2a(\boldsymbol{v}^n)$ and, between the two solutions, the one with a smaller objective value is optimal to PG.

A.4 Link-Based User Equilibrium Conditions under Nonlinear Pricing

This section investigates properties under which equilibrium conditions and the UE problem can be formulated using link flows. Below, Section 8.4.1 discusses one such property that relies on the relationship between $SUB1(\boldsymbol{v}^n)$ and its dual problem. (Recall that $SUB1(v^n)$ is a problem associated with the PG problem in Step 2 of SD.) Then, Section 8.4.2 provides two sets of link-based UE conditions. One is equivalent to (A.2.3) – (A.2.6) when $SUB1(\boldsymbol{v}^n)$ has no duality gap and the other is only sufficient. In Section 8.4.3, we show that equilibrium conditions and the UE problem under area-based and two-part pricing schemes can be stated in terms of link flows.

A.4.1 Lagrangian Dual Problems

In this and subsequent sections, we remove the iteration index, n, from $SUB(\boldsymbol{v}^n)$ because it is irrelevant. The problem is well defined for any $\boldsymbol{v}$ such that, for some travel demand vector $\boldsymbol{d}$, $(\boldsymbol{v}, \boldsymbol{d}) \in V^f$.

For a given OD pair k, the Lagrangian dual problem for $SUB1(\boldsymbol{v})$ can be written as follows (see, e.g., Bazaraa et al., 2006):

$$\begin{aligned} D1(\boldsymbol{v}): \quad & max \quad L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}}) \\ s.t. \quad & \alpha_1^k, \alpha_2^k \geq 0 \end{aligned} \tag{A.4.1}$$

where $L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}})$, the Lagrangian function associated with $SUB1(\boldsymbol{v})$, is defined as follows:

$$\begin{aligned} L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}}) = \quad & min \quad \sum_{a \in \Omega} s_a(\boldsymbol{v}) z_a^k + w^k + \sum_{m=1}^{2} \alpha_m^k \left(\beta_m + \mu_m \sum_{a \in \Omega_1} l_a z_a^k - w^k \right) \\ & s.t. \quad A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & \qquad z_a^k \in \{0,1\}, \forall a \in \Omega. \end{aligned} \tag{A.4.2}$$

The variables α_m^k, for $m = 1\,,2$, are Lagrange multipliers constrained to be nonnegative. In literature, some refer to the above problem as a Lagrangian subproblem.

Let $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ and $(\bar{\alpha}_1^k, \bar{\alpha}_2^k)$ solve $SUB1(\boldsymbol{v})$ and $D1(\boldsymbol{v})$, respectively. Then, it follows from the weak duality theorem (see, Bazaraa et al., 2006) that:

$$\sum_{a \in \Omega} s_a(\boldsymbol{v}) \hat{z}_a^k + \widehat{w}^k \geq L_v^k\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right). \tag{A.4.2}$$

The result below assumes that the inequality in the above expression holds at equality, i.e., the strong duality condition holds or $SUB1(\boldsymbol{v})$ has no or zero duality gap.

Lemma A.1: If $SUB1(\boldsymbol{v})$ has no duality gap, then its solution also solves the Lagrangian subproblem of $D1(\boldsymbol{v})$.

Proof: As discussed above, let $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ and $(\bar{\alpha}_1^k, \bar{\alpha}_2^k)$ solve $SUB1(\boldsymbol{v})$ and $D1(\boldsymbol{v})$, respectively. Then, the following must hold:

$$\begin{aligned}
\sum_{a \in \Omega} s_a(\boldsymbol{v}) \hat{z}_a^k + \widehat{w}^k \quad & = L_v^k\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right) \\
& = \min\left\{\sum_{a \in \Omega} s_a(\boldsymbol{v}) z_a^k + w^k + \sum_{m=1}^{2} \bar{\alpha}_m^k \left(\beta_m + \mu_m \sum_{a \in \Omega_1} l_a z_a^k - w^k\right) : A\boldsymbol{z}^k = \boldsymbol{E}_k, z_a^k \in \{0,1\}\right\} \\
& \leq \sum_{a \in \Omega} s_a(\boldsymbol{v}) \hat{z}_a^k + \widehat{w}^k + \sum_{m=1}^{2} \bar{\alpha}_m^k \left(\beta_m + \mu_m \sum_{a \in \Omega_1} l_a \hat{z}_a^k - \widehat{w}^k\right) \\
& \leq \sum_{a \in \Omega} s_a(\boldsymbol{v}) \hat{z}_a^k + \widehat{w}^k.
\end{aligned} \tag{A.4.3}$$

In the above, the first two equalities follow from the zero duality gap assumption and the definition of the Lagrangian function at the optimal dual solution $\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right)$, respectively. Next, the first inequality holds because $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ is feasible to the minimization problem. When viewed as an optimal solution to $SUB1(\boldsymbol{v})$, $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ satisfies $\left(\beta_m + \mu_m \sum_{a \in \Omega_1} l_a \hat{z}_a^k - \widehat{w}^k\right) \leq 0$ for $m = 1{,}2$. Combining the latter with the fact that $\bar{\alpha}_m^k \geq 0$, for $m = 1, 2$, implies that $\sum_{m=1}^{2} \bar{\alpha}_m^k \left(\beta_m + \mu_m \sum_{a \in \Omega_1} l_a \hat{z}_a^k - \widehat{w}^k\right) \leq 0$. Thus, the last inequality must hold.

The above sequence of equalities and inequalities begins and ends with the same expression. Thus, the two inequalities must be equalities, i.e., $(\hat{\boldsymbol{z}}^k, \widehat{w}^k)$ must be optimal to the minimization problem, i.e., the Lagrangian subproblem of $D1(\boldsymbol{v})$ associated with $\left(\bar{\alpha}_1^k, \bar{\alpha}_2^k\right)$. □

To make a problem structure more evident, observe that β_m and α_m^k, $m = 1, 2$, are constants with respect to the minimization and the Lagrangian subproblem can be written as

$$L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}}) = (\alpha_1^k \beta_1 + \alpha_2^k \beta_2) + \begin{array}{ll} \min & \sum_{a \in \Omega} s_a(\boldsymbol{v}) z_a^k + w^k(1 - \alpha_1^k - \alpha_2^k) + (\alpha_1^k \mu_1 + \alpha_2^k \mu_2) \sum_{a \in \Omega_1} l_a z_a^k \\ s.t. & A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & z_a^k \in \{0,1\}, \forall a \in \Omega. \end{array} \tag{A.4.4}$$

In the above, w^k is unrestricted. When $\alpha_1^k + \alpha_2^k > 1$, $L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}}) = -\infty$ because setting $w^k = \infty$ is optimal. On the other hand, when $\alpha_1^k + \alpha_2^k \leq 1$, the optimal value for w^k is zero and $L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}})$ is finite. To maximize the value of $L_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}})$ in problem $D1(\boldsymbol{v})$, it makes sense to restrict α_1^k and α_2^k to the region where $\alpha_1^k + \alpha_2^k \leq 1$ and $\alpha_1^k, \alpha_2^{\boldsymbol{k}} \geq 0$. Thus, the Lagrangian dual problem for $SUB1(\boldsymbol{v})$ can be equivalently written as:

$$D2(\boldsymbol{v}): \begin{array}{ll} \max & \tilde{L}_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}}) + (\alpha_1^k \beta_1 + \alpha_2^k \beta_2) \\ s.t. & \alpha_1^k + \alpha_2^{\boldsymbol{k}} \leq 1 \\ & \alpha_1^k, \alpha_2^{\boldsymbol{k}} \geq 0 \end{array} \tag{A.4.5}$$

where $\tilde{L}_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}})$ is a modified Lagrangian function and, because A is totally unimodular, it can defined as follows:

$$\tilde{L}_v^k(\alpha_1^k, \alpha_2^{\boldsymbol{k}}) = \begin{array}{ll} \min & \sum_{a \in \Omega^1} \left(s_a(\boldsymbol{v}) + (\alpha_1^k \mu_1 + \alpha_2^k \mu_2) l_a\right) z_a^k + \sum_{a \in \Omega^2} s_a(\boldsymbol{v}) z_a^k \\ s.t. & A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & z_a^k \geq 0, \forall a \in \Omega \end{array} \tag{A.4.6}$$

We also refer to the problem directly above as the modified Lagrangian subproblem. Because $D1(\boldsymbol{v})$ and $D2(\boldsymbol{v})$ are equivalent, it follows from Lemma 4.1 that, if $SUB1(\boldsymbol{v})$ has no duality gap, its solution also solves the modified Lagrangian subproblem and

$$\sum_{a \in \Omega} s_a(\boldsymbol{v}) \hat{z}_a^k + \widehat{w}^k = \tilde{L}_v^k(\bar{\alpha}_1^k, \bar{\alpha}_2^k) + (\bar{\alpha}_1^k \beta_1 + \bar{\alpha}_2^k \beta_2). \tag{A.4.7}$$

A.4.2 Link-Based Equilibrium Conditions: General Case

For a given $(\boldsymbol{f}, \boldsymbol{d}) \in V^f$, define

$$\begin{aligned} x_a^k(\boldsymbol{f}) &= \sum\nolimits_{a \in \Omega} \sum\nolimits_{r \in TP^k} \delta_{ar} f_r^k \\ y_a^k(\boldsymbol{f}) &= \sum\nolimits_{a \in \Omega} \sum\nolimits_{r \in NP^k} \delta_{ar} f_r^k \\ \sigma_k &= \sum\nolimits_{r \in TP^k} f_r^k \\ \eta_k &= \sum\nolimits_{r \in NP^k} f_r^k \end{aligned} \tag{A.4.8}$$

In words, $\boldsymbol{x}^k(\boldsymbol{f})$ and $\boldsymbol{y}^k(\boldsymbol{f})$ are, respectively, vectors of link flows on toll and toll-free paths associated with $(\boldsymbol{f}, \boldsymbol{d})$. As constructed, $y_a^k(\boldsymbol{f}) = 0, \forall a \in \Omega^1, k \in K$, i.e., $\boldsymbol{y}(\boldsymbol{f})$ is the link-flow vector associated with toll-free paths. In the last two equations, σ_k and η_k are variables representing the numbers of users who pay and do not pay tolls for OD pair k, respectively. For every OD pair k, the above vectors and variables satisfy the following linear systems:

$$\begin{aligned} [A_1 : A_2]\boldsymbol{x}^k &= \sigma_k \boldsymbol{E}_k \\ [0 : A_2]\boldsymbol{y}^k &= \eta_k \boldsymbol{E}_k \\ \sigma_k + \eta_k &= d_k \end{aligned} \quad \text{(A.4.10)}$$

where, as previously defined, A_1 and A_2 are node-arc incidence matrices for subnetworks induced by arcs in the sets Ω^1 and Ω^2, respectively.

The above motivates a link-based representation of feasible flow-demand pairs based on $\boldsymbol{x}^k$ and $\boldsymbol{y}^k$. In particular, the set of all feasible flow-demand pair can be equivalently written as

$$V^x = \left\{ \begin{array}{r} (\boldsymbol{v}, \boldsymbol{d}) : \boldsymbol{v} = \sum_{k \in K} (\boldsymbol{x}^k + \boldsymbol{y}^k), d_k = \sigma_k + \eta_k, A\boldsymbol{x}^k = \sigma_k \boldsymbol{E}_k, [0 : A_2]\boldsymbol{y}^k = \eta_k \boldsymbol{E}_k, \\ \boldsymbol{x}^k \geq 0, \boldsymbol{y}^k \geq 0, \sigma_k \geq 0, \eta_k \geq 0, \forall k \in K \end{array} \right\}. \quad \text{(A.4.11)}$$

Because the value of $y_a^k, \forall a \in \Omega^1$, is unspecified in the above expression, it is assumed that they are always zero, i.e., flows associated with $\boldsymbol{y}$ do not enter the tolling area. Later, we also refer to elements of V^x in a disaggregate form or as a quadruplet $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{\eta}) \in V^x$, i.e., we also define V^x as follows:

$$V^x = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{\eta}) : A\boldsymbol{x}^k = \sigma_k \boldsymbol{E}_k, [0 : A_2]\boldsymbol{y}^k = \eta_k \boldsymbol{E}_k, \boldsymbol{x}^k \geq 0, \boldsymbol{y}^k \geq 0, \sigma_k \geq 0, \eta_k \geq 0, \forall k \in K\}. \quad \text{(A.4.12)}$$

For every $(\boldsymbol{v}, \boldsymbol{d})$ in V^x, there must exist a pair $\big(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\big)$, not necessarily unique, such that $x_a^k = \sum_{a \in \Omega} \sum_{r \in TP^k} \delta_{ar} f_r^k(\boldsymbol{v})$, $y_a^k = \sum_{a \in \Omega} \sum_{r \in NP^k} \delta_{ar} f_r^k(\boldsymbol{v})$, and $d_k(\boldsymbol{v}) = \sum_{r \in TP^k} f_r^k(\boldsymbol{v}) + \sum_{r \in NP^k} f_r^k(\boldsymbol{v})$. Moreover, the pair $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ also belongs to V^f and such a pair is said to be compatible with $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$. The theorem below specifies conditions for equilibrium based on elements in V^x or link flows. Its proof relies on the zero duality gap assumption and the above relationship between V^x and V^f.

Theorem A.2: Assume that $SUB1(\boldsymbol{v})$ has no duality gap. Then, $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$ is in tolled UE if and only if, for each $k \in K$, there exist $\boldsymbol{\rho}^k \in R^N$, $\boldsymbol{\gamma}^k \in R^N$, α_1^k, and α_2^k such that the following link-based conditions hold:

$$\left(\alpha_1^k\mu_1 + \alpha_2^k\mu_2\right)l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega^1, k \in K: x_{ij}^k > 0 \quad (A.4.13)$$
$$\left(\alpha_1^k\mu_1 + \alpha_2^k\mu_2\right)l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega^1, k \in K: x_{ij}^k = 0 \quad (A.4.14)$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega^2, k \in K: x_{ij}^k > 0 \quad (A.4.15)$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega^2, k \in K: x_{ij}^k = 0 \quad (A.4.16)$$
$$\left(\alpha_1^k\beta_1 + \alpha_2^k\beta_2\right) + \boldsymbol{E}_k^T\boldsymbol{\rho}^k = D_k^{-1}(\sigma_k + \eta_k), \quad \forall k \in K: \boldsymbol{x}^k \neq 0 \quad (A.4.17)$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) = 0, \quad \forall (i,j) \in \Omega^2, k \in K: y_{ij}^k > 0 \quad (A.4.18)$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega^2, k \in K: y_{ij}^k = 0 \quad (A.4.19)$$
$$\boldsymbol{E}_k^T\boldsymbol{\gamma}^k = D_k^{-1}(\sigma_k + \eta_k), \quad \forall k \in K: \boldsymbol{y}^k \neq 0 \quad (A.4.20)$$
$$(\alpha_1^k, \alpha_2^k) \text{ solves } D2(\boldsymbol{v}), \quad \forall k \in K \quad (A.4.21)$$

Proof: For each $k \in K$, assume that there exist $\boldsymbol{\rho}^k, \boldsymbol{\gamma}^k, \alpha_1^k$, and α_2^k satisfying conditions (A.4.13) – (A.4.21). Below, we show that, for every OD pair $k \in K$, the generalized cost of all utilized routes, toll or toll-free, equal $D_k^{-1}(d_k)$ and the costs of those not utilized are at least as large. Consider a toll-free route that is utilized with respect to any pair $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ compatible with $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$, i.e., $r \in NP_{++}^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$. If $\delta_{ar} = 1$, then there must be flows on link a, i.e., $y_a^k(\boldsymbol{v}) > 0$. Summing together expression (A.4.18) for all a such that $\delta_{ar} = 1$ yields

$$0 = \sum_{(i,j)\in\Omega} \delta_{(i,j)r}\left(s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right)\right) = \sum_{a\in\Omega} \delta_{ar}s_a(\boldsymbol{v}) - \gamma_{o(k)}^k + \gamma_{d(k)}^k = \sum_{a\in\Omega} \delta_{ar}s_a(\boldsymbol{v}) - \boldsymbol{E}_k^T\boldsymbol{\gamma}^k \quad \text{(A.4.22)}$$

where $o(k)$ and $d(k)$ denote, respectively, the origin and destination of OD pair k. Thus, $\sum_{a\in\Omega} \delta_{ar}s_a(\boldsymbol{v}) = \boldsymbol{E}_k^T\boldsymbol{\gamma}^k$ and it follows from (A.4.20) that $\sum_{a\in\Omega} \delta_{ar}s_a(\boldsymbol{v}) = D_k^{-1}(\sigma_k + \eta_k) = D_k^{-1}(d_k)$. Thus, the cost of path r equals the value of the inverse demand function at the realized demand d_k. When a toll-free route r is not utilized, i.e., $r \in NP_0^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$, some link on route r has no flow, i.e., $y_a^k(\boldsymbol{v}) = 0$ for some a such that $\delta_{ar} = 1$. For arcs satisfying the latter, (A.4.19) indicates that $s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) \geq 0$ and the following holds:

$$\begin{aligned} 0 \leq \sum_{(i,j)\in\Omega} \delta_{(i,j)r}\left(s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right)\right) &= \sum_{a\in\Omega} \delta_{ar}s_a(\boldsymbol{v}) - \gamma_{o(k)}^k + \gamma_{d(k)}^k \\ &= \sum_{a\in\Omega} \delta_{ar}s_a(\boldsymbol{v}) - \boldsymbol{E}_k^T\boldsymbol{\gamma}^k \end{aligned} \quad \text{(A.4.23)}$$

From above, $\sum_{a\in\Omega} \delta_{ar}s_a(\boldsymbol{v}) \geq \boldsymbol{E}_k^T\boldsymbol{\gamma}^k = D_k^{-1}(d_k)$, i.e., the cost of a nonutilized toll-free path cannot be smaller than the value of the inverse demand function. Thus, among the toll-free paths, the utilized ones have costs equal to $D_k^{-1}(d_k)$ and those not utilized cannot have a lower cost.

For a toll route $r \in TP_{++}^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$, let $z_a^k = \delta_{ar}, \forall a \in \Omega$. As constructed, $\boldsymbol{z}^k$ is feasible to the modified Lagrangian subproblem associated with $\tilde{L}_v^k\left(\alpha_1^k, \alpha_2^{\boldsymbol{k}}\right)$ at the end of Section 8.4.1. The dual of this subproblem can be written as follows:

$$\begin{Bmatrix} max & \boldsymbol{E}_k^t \boldsymbol{\rho}^{\boldsymbol{k}} & \\ s.t. & \rho_i^k - \rho_j^k \le (\alpha_1\mu_1 + \alpha_2\mu_2)l_{ij} + s_{ij}(\boldsymbol{v}), & \forall(i,j) \in \Omega^1 \\ & \rho_i^k - \rho_j^k \le s_{ij}(\boldsymbol{v}), & \forall(i,j) \in \Omega^2 \\ & \boldsymbol{\rho}^{\boldsymbol{k}} \text{ unresticted} & \end{Bmatrix} \tag{A.4.25}$$

The hypothesis that $\boldsymbol{\rho}^k, \boldsymbol{\gamma}^k, \alpha_1^k$, and α_2^k exist ensures that the above dual problem has a solution. Then, it follows from the strong duality theorem in linear programming (see, e.g., Bazaraa et al., 2010) that $\tilde{L}_v^k(\alpha_1^k, \alpha_2^k) = \boldsymbol{E}_k^T \boldsymbol{\rho}^k$ and both $\boldsymbol{z}^k$ and $\boldsymbol{\rho}^k$ are optimal to their respective problems. Because $D1(\boldsymbol{v})$ and $D2(\boldsymbol{v})$ are equivalent and $SUB1(\boldsymbol{v})$ has no duality gap, $\tilde{L}_v^k(\alpha_1^k, \alpha_2^k) = \boldsymbol{E}_k^T \boldsymbol{\rho}^k$ and the following holds:

$$\begin{aligned} \sum_{a \in \Omega} s_a(\boldsymbol{v}) \boldsymbol{z}_a^k + w^k &= L_v^k(\alpha_1^k, \alpha_2^k) = \tilde{L}_v^k(\alpha_1^k, \alpha_2^k) + (\alpha_1^k \beta_1 + \alpha_2^k \beta_2) \\ &= \boldsymbol{E}_k^T \boldsymbol{\rho}^k + (\alpha_1^k \beta_1 + \alpha_2^k \beta_2). \end{aligned} \tag{A.4.26}$$

In the above, $w^k = \max\{\beta_1 + \mu_1 \sum_{a \in \Omega^1} l_a z_a^k,\ \beta_2 + \mu_2 \sum_{a \in \Omega^1} l_a z_a^k\} = T^{max}(\sum_{a \in \Omega^1} l_a z_a^k)$ because $(\boldsymbol{z}^k, w^k)$ is optimal to $SUB1(\boldsymbol{v})$. Replacing w^k with $T^{max}(\sum_{a \in \Omega^1} l_a z_a^k)$ in the preceding equation yields

$$\sum_{a \in \Omega} s_a(\boldsymbol{v}) \boldsymbol{z}_a^k + T^{max}\left(\sum\nolimits_{a \in \Omega^1} l_a z_a^k,\right) = \boldsymbol{E}_k^T \boldsymbol{\rho}^k + (\alpha_1^k \beta_1 + \alpha_2^k \beta_2) = D_k^{-1}(d_k), \tag{A.4.27}$$

where the last equality follows from (A.4.17). Thus, the cost of a utilized toll path equals $D_k^{-1}(d_k)$.

When $r \in TP_0^k(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$, letting $z_a^k = \delta_{ar}, \forall a \in \Omega$, may not yield an optimal solution to the modified Lagrangian subproblem. When the path is not utilized, $x_{ij}^k(\boldsymbol{v})$ may equal zero when $z_{ij}^k = 1$. For such link (i,j), (A.4.14) and (A.4.16) imply that $z_{ij}^k\left((\alpha_1^k \mu_1 + \alpha_2^k \mu_2) l_{ij} + s_{ij}(\boldsymbol{v}) - \rho_i^k + \rho_j^k\right) \ge 0$ and $z_{ij}^k\left(s_{ij}(\boldsymbol{v}) - \rho_i^k + \rho_j^k\right) \ge 0$, i.e., the complementary slackness condition may not hold and $\boldsymbol{z}^k$ may not solve the modified Lagrangian subproblem at the end of Section 8.4.1. However, because $\boldsymbol{z}^k$ is still feasible to the subproblem, the weak duality theorem applies and

$$\sum_{a \in \Omega^1} \left(s_a(\boldsymbol{v}) + (\alpha_1^k \mu_1 + \alpha_2^k \mu_2) l_a\right) z_a^k + \sum_{a \in \Omega^2} s_a(\boldsymbol{v}) z_a^k \ge \tilde{L}_v^k(\alpha_1^k, \alpha_2^k) = \boldsymbol{E}_k^T \boldsymbol{\rho}^k. \tag{A.4.28}$$

$(\alpha_1^k \beta_1 + \alpha_2^k \beta_2)$ to both sides of the above and using (A.4.17) yields

$$\sum_{a \in \Omega} s_a(\boldsymbol{v})\, z_a^k + \sum_{m=1}^{2} \alpha_m^k \left(\beta_m + \mu_m \sum_{a \in \Omega_1} l_a z_a^k\right) \ge \boldsymbol{E}_k^T \boldsymbol{\rho}^k + (\alpha_1^k \beta_1 + \alpha_2^k \beta_2) = D_k^{-1}(d_k). \tag{A.4.29}$$

Since $\max\{\beta_1 + \mu_1 \sum_{a \in \Omega^1} l_a z_a^k, \beta_2 + \mu_2 \sum_{a \in \Omega^1} l_a z_a^k\} \ge \sum_{m=1}^2 \alpha_m^k (\beta_m + \mu_m \sum_{a \in \Omega_1} l_a z_a^k)$ when $\alpha_1^k + \alpha_2^k = 1$ and $\alpha_1^k, \alpha_2^k \ge 0$, it follows from above that

$$\sum_{a\in\Omega} s_a(\boldsymbol{v})\boldsymbol{z}_a^k + T^{max}\left(\sum\nolimits_{a\in\Omega^1} l_a z_a^k,\right) \geq \sum_{a\in\Omega} s_a(\boldsymbol{v})\, z_a^k + \sum_{m=1}^{2} \alpha_m^k \left(\beta_m + \mu_m \sum_{a\in\Omega_1} l_a z_a^k\right) \geq D_k^{-1}(d_k). \tag{A.4.30}$$

Thus, if a toll path is not utilized, its cost is no smaller than $D_k^{-1}(d_k)$. Finally, it follows from (A.4.22), (A.4.23), and (A.4.30) that any pair $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ compatible with $(\boldsymbol{v}, \boldsymbol{d})$ is in tolled UE.

For the converse, assume that the flows on toll and toll-free paths are in tolled UE. For $r \in TP_{++}^k\big(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\big)$, $z_a^k = \delta_{ar}, \forall a \in \Omega$, must solve $SUB1(\boldsymbol{v})$ because path r must be one with the least generalized cost by definition. The zero duality gap assumption and Lemma A.1 imply that $\boldsymbol{z}^k$ also solves the modified Lagrangian subproblem at the end of Section 8.4.1. Then, it is easy to show that the optimal dual vector, $\boldsymbol{\rho}^k$, associated with the subproblem satisfies (A.4.13) – (A.4.17) with (α_1^k, α_2^k) as specified in (A.4.21). The similar also holds with $r \in NP_{++}^k\big(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\big)$, $\boldsymbol{\gamma}^k$, $SUB2a(\boldsymbol{v})$ and (A.4.18) – (A.4.20).□

There are also link-based equilibrium conditions without relying on the zero duality gap assumption. Typically, they are only sufficient. For example, the theorem below provides a set of such conditions. Unlike the previous theorem, there are two set of node potentials, $\boldsymbol{\rho}^k$ and $\boldsymbol{\psi}^k$, for the link flows x_{ij}^k.

Theorem A.3: A pair $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$ is in tolled UE if there exist $\boldsymbol{\rho}^k, \boldsymbol{\psi}^k$, and $\boldsymbol{\gamma}^k$ such that following link-based conditions hold:

$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k > 0 \tag{A.4.31}$$
$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k = 0 \tag{A.4.32}$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k > 0 \tag{A.4.33}$$
$$s_{ij}(\boldsymbol{v}) - \left(\rho_i^k - \rho_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k = 0 \tag{A.4.34}$$
$$\mu_2 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) = 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k > 0 \tag{A.4.35}$$
$$\mu_2 l_{ij} + s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k = 0 \tag{A.4.36}$$
$$s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k > 0 \tag{A.4.37}$$
$$s_{ij}(\boldsymbol{v}) - \left(\psi_i^k - \psi_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k = 0 \tag{A.4.38}$$
$$max\{\beta_1 + \boldsymbol{E}_k^T \boldsymbol{\rho}^k, \beta_2 + \boldsymbol{E}_k^T \boldsymbol{\psi}^k\} = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{x}^k \neq 0 \tag{A.4.39}$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k > 0 \tag{A.4.40}$$
$$s_{ij}(\boldsymbol{v}) - \left(\gamma_i^k - \gamma_j^k\right) \geq 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k = 0 \tag{A.4.41}$$
$$\boldsymbol{E}_k^T \boldsymbol{\gamma}^k = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{y}^k \neq 0 \tag{A.4.42}$$

Proof: For any $\left(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\right)$ compatible with $(\boldsymbol{v}, \boldsymbol{d})$ and $r \in TP_{++}^{k}\left(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v})\right)$, it follows from arguments similar to those in Theorem A.2 that (A.4.31), (A.4.33), (A.4.35), and (A.4.37) lead to the following:

$$\begin{aligned} \mu_1 \sum_{a\in\Omega^1} \delta_{ar} l_a + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}) &= \boldsymbol{E}_k^T \boldsymbol{\rho}^k \\ \mu_2 \sum_{a\in\Omega^1} \delta_{ar} l_a + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}) &= \boldsymbol{E}_k^T \boldsymbol{\psi}^k \end{aligned} \tag{A.4.43}$$

Substituting the above expressions for $\boldsymbol{E}_k^T \boldsymbol{\rho}^k$ and $\boldsymbol{E}_k^T \boldsymbol{\psi}^k$ into (A.4.22) yields

$$\begin{aligned} & max\left\{\beta_1 + \mu_1 \sum_{a\in\Omega^1} \delta_{ar} l_a + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}), \beta_2 + \mu_2 \sum_{a\in\Omega^1} \delta_{ar} l_a + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v})\right\} \\ & \qquad = D_k^{-1}(d_k) \\ & max\left\{\beta_1 + \mu_1 \sum_{a\in\Omega^1} \delta_{ar} l_a, \beta_2 + \mu_2 \sum_{a\in\Omega^1} \delta_{ar} l_a\right\} + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(d_k) \\ & T^{max}\left(\sum_{a\in\Omega^1} \delta_{ar} l_a\right) + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}) = D_k^{-1}(d_k) \end{aligned} \tag{A.4.44}$$

Similarly, the following must hold

$$\begin{aligned} T^{max}\left(\sum_{a\in\Omega^1} \delta_{ar} l_a\right) + \sum_{a\in\Omega} \delta_{ar} s_a(\boldsymbol{v}) &\geq D_k^{-1}(d_k), \forall r \in TP_0^k \\ \sum_{a\in\Omega^2} \delta_{ar} s_a(\boldsymbol{v}) &= D_k^{-1}(d_k), \forall r \in NP_{++}^k \\ \sum_{a\in\Omega^2} \delta_{ar} s_a(\boldsymbol{v}) &\geq D_k^{-1}(d_k), \forall r \in NP_0^k \end{aligned} \tag{A.4.45}$$

Then, the last four equations imply that the costs for all utilized paths, toll-free or otherwise, equal $D_k^{-1}(d_k)$ and the costs of those not utilized cannot be lower, i.e., the tolled equilibrium conditions hold for any $(\boldsymbol{f}(\boldsymbol{v}), \boldsymbol{d}(\boldsymbol{v}))$ compatible with $(\boldsymbol{v}, \boldsymbol{d})$. □

A.4.3 Link-Based Equilibrium Conditions: Two-part Pricing

This section considers two special cases in nonlinear pricing: area-based and two-part pricing. Mathematically, the latter corresponds to setting β_2 and μ_2 in the tolling function to zero. Doing so yields $T^{max}(\ell) = \beta_1 + \mu_1 \ell$ (see case (a) in Figure 2.2). Additionally, if $\mu_1 = 0$, then $T^{max}(\ell) = \beta_1$ and two-part pricing reduces to area-based pricing. The results below demonstrate that $SUB1(v)$ has no duality gap and provide link-based UE conditions for two-part pricing.

Lemma A.4: If $T^{max}(\ell) = \beta_1 + \mu_1 \ell$, where β_1 and μ_1 are both nonnegative, then $SUB1(\boldsymbol{v})$ has no duality gap.

Proof: For $T^{max}(\ell)$ as given, $SUB1(\boldsymbol{v})$ reduces to

$$\begin{aligned} SUB1(\boldsymbol{v}): \quad min \quad & \sum_{a\in\Omega} s_a(\boldsymbol{v})z_a^k + w^k \\ s.t. \quad & A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & \beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k \le w^k \\ & z_a^k \in \{0,1\}, \forall a \in \Omega. \end{aligned} \tag{A.4.46}$$

The Lagrangian dual problem (or $D1(\boldsymbol{v})$) of the above can be written as follows:

$$\max\{L_v^k(\alpha_1^k): 0 \le \alpha_1^k \le 1\} \tag{A.4.47}$$

where $L_v^k(\alpha_1^k) = \alpha_1^k\beta_1 + \min\{\sum_{a\in\Omega} s_a(\boldsymbol{v})z_a^k + \alpha_1^k\mu_1 \sum_{a\in\Omega^1} l_a z_a^k : A\boldsymbol{z}^k = \boldsymbol{E}_k, z_a^k \ge 0\}$. As before, we can replace the binary restriction with $z_a^k \ge 0$ because A is totally unimodular. Observe that the second constraint in $SUB1(\boldsymbol{v})$ must be hold at equality, i.e., $\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k = w^k$ in order to minimize the objective function. Thus, w^k in the objective of $SUB1(\boldsymbol{v})$ can be replaced by $\beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k$ and the problem can be written as

$$\begin{aligned} SUB1(\boldsymbol{v}): \quad min \quad & \sum_{a\in\Omega} s_a(\boldsymbol{v})z_a^k + \beta_1 + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k \\ s.t. \quad & A\boldsymbol{z}^k = \boldsymbol{E}_k \\ & z_a^k \ge 0, \forall a \in \Omega. \end{aligned} \tag{A.4.48}$$

Comparing the two equivalent forms of $SUB1(\boldsymbol{v})$ yields that

$$\min\left\{\sum_{a\in\Omega} s_a(\boldsymbol{v})z_a^k + w^k\right\} = \min\left\{\sum_{a\in\Omega} s_a(\boldsymbol{v})z_a^k + \mu_1 \sum_{a\in\Omega^1} l_a z_a^k\right\} + \beta_1 = L_v^k(1) \tag{A.4.49}$$

Thus, $\alpha_1^k = 1$ is optimal to the Lagrangian dual problem and the objective values of $SUB1(\boldsymbol{v})$ and its Lagrangian dual problem are the same, i.e., there is no duality gap. □

Theorem A.5: Let $T^{max}(\ell) = \beta_1 + \mu_1\ell$. Then, a pair $(\boldsymbol{v}, \boldsymbol{d}) \in V^x$ is in tolled UE if and only if there exist $\boldsymbol{\rho}^k$ and $\boldsymbol{\eta}^k$ such the following link-based conditions hold:

$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - (\rho_i^k - \rho_j^k) = 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k > 0 \quad (A.4.50)$$

$$\mu_1 l_{ij} + s_{ij}(\boldsymbol{v}) - (\rho_i^k - \rho_j^k) \ge 0, \quad \forall (i,j) \in \Omega_1, k \in K: x_{ij}^k = 0 \quad (A.4.51)$$

$$s_{ij}(\boldsymbol{v}) - (\rho_i^k - \rho_j^k) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k > 0 \quad (A.4.52)$$

$$s_{ij}(\boldsymbol{v}) - (\rho_i^k - \rho_j^k) \ge 0, \quad \forall (i,j) \in \Omega_2, k \in K: x_{ij}^k = 0 \quad (A.4.53)$$

$$\beta_1 + \boldsymbol{E}_k^T\boldsymbol{\rho}^k = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{x}^k \ne 0 \quad (A.4.54)$$

$$s_{ij}(\boldsymbol{v}) - (\gamma_i^k - \gamma_j^k) = 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k > 0 \quad (A.4.55)$$

$$s_{ij}(\boldsymbol{v}) - (\gamma_i^k - \gamma_j^k) \ge 0, \quad \forall (i,j) \in \Omega_2, k \in K: y_{ij}^k = 0 \quad (A.4.56)$$

$$\boldsymbol{E}_k^T\boldsymbol{\gamma}^k = D_k^{-1}(d_k), \quad \forall k \in K, \boldsymbol{y}^k \ne 0 \quad (A.4.57)$$

Proof: The result follows directly from Theorem A.2 and Lemma 5.5. When applying Theorem A.2 to the case where $T^{max}(\ell) = \beta_1 + \mu_1 \ell$, observe that there is no α_2^k. In addition, the argument in Lemma 5.5 shows that $\alpha_1^k = 1$ solves $D2(\boldsymbol{v})$ in condition (A.4.21) of Theorem A.2. □

Observe that (A.4.50) – (A.4.57) are the KKT conditions of the following optimization problem or the tolled UE problem under two-part pricing (TUE2):

$$
\begin{aligned}
TUE2: \quad min \quad & \sum_{a \in \Omega^1} \int_0^{\sum_k x_a^k} s_a(z)dz + \sum_{a \in \Omega^2} \int_0^{\sum_k x_a^k + y_a^k} s_a(z)dz - \sum_{k \in K} \int_0^{\sigma_k + \eta_k} D_k^{-1}(z)dz \\
& + \beta_1 \sum_{k \in K} \sigma_k + \mu_1 \sum_{a \in \Omega^1} \sum_{k \in K} l_a x_a^k \\
s.t. \quad & [A_1 : A_2] x^k - E_k \sigma_k = 0, \forall k \in K \\
& [0 : A_2] y^k - E_k \eta_k = 0, , \forall k \in K \\
& x^k, y^k, \sigma_k, \eta_k \geq 0, , \forall k \in K
\end{aligned}
\tag{A.4.58}
$$

In the objective, the first three terms are convex functions and represent the objective function of a problem for finding a (toll-free) UE flow-demand pair when demands are elastic. The last two terms determine the total toll collected in two parts, the access and VMT fee. The first two constraints are flow-balance constraints for users who pay, σ_k, and do not pay toll, η_k. By letting $\boldsymbol{\rho}^k$ and $\boldsymbol{\gamma}^k$ be the multiplier vectors associated with the first two constraints, it is straightforward to show that the KKT conditions of the above problem reduce to conditions (A.4.26) – (A.4.33). Thus, the solution to the above problem yields a UE flow-demand pair $(\boldsymbol{v}, \boldsymbol{d})$ under two-part pricing, where $\boldsymbol{v} = \sum_{k \in K} \boldsymbol{x}^k + \boldsymbol{y}^k$ and $d_k = \sigma_k + \eta_k$.

As stated above, $TUE2$ involves no path flow (or f_r^k) and is a linearly constrained convex program, a problem that can be solved by commercial software such as CONOPT (see, e.g., Drud, 1992). To illustrate that standard algorithms in the literature with some modifications are applicable to $TUE2$, we state the Frank-Wolfe algorithm as it applies to $TUE2$ in the Appendix.

A.5 Finding Optimal Nonlinear Tolling Schemes

For the tolling function based on $T^{max}(\cdot)$, the problem of finding an optimal nonlinear tolling scheme can be formulated as follows:

$$
\begin{aligned}
NLT: \quad \max \quad & \sum_{k \in K} \int_0^{d_k} D_k^{-1}(\chi) d\chi - s(\boldsymbol{v})^T \boldsymbol{v} \\
\text{s.t.} \quad & \text{restrictions on } \beta_1, \beta_2, \mu_1, \text{and } \mu_2 \\
& (\boldsymbol{v}, \boldsymbol{d}) \in V^f \\
& (\boldsymbol{v}, \boldsymbol{d}) \text{ satisfies } (3.1) - (3.4)
\end{aligned}
\tag{A.5.1}
$$

of the above is to maximize the social benefit. In the constraints, restrictions on the four pricing parameters depend on the pricing function of interest. For example, setting β_2, μ_1, and μ_2 to zero and allowing β_1 to be in the interval $[0, \beta_1^{max}]$ yield an area-based pricing scheme. On the other hand, setting β_2 and μ_2 to zero and allowing β_1 and μ_1 to be in the intervals $[0, \beta_1^{max}]$ and $[0,\ \mu_1^{max}]$, respectively, would generate a two-part pricing scheme instead. The remaining constraints ensure that the flow-demand pair is feasible and satisfies the tolled UE conditions. In words, NLT finds a set of pricing parameters such that the associated UE flow-demand pair yields the maximum social benefit.

As stated, NLT is a mathematical program with equilibrium constraints (see, e.g., Luo et al., 1996), a class of optimization problems generally difficult to solve. However, NLT contains at most four main decision variables–the pricing parameters. The other variables $(\boldsymbol{v}, \boldsymbol{d})$ react to or are induced by the pricing parameters via the last two set of constraints. As such, NLT can be solved approximately using a coordinate search technique (see, e.g., Bazaraa et al., 2006), one that sequentially searches for an optimal solution one decision variable (or coordinate) at a time. Because the feasible region of NLT is not convex, search and other algorithms in nonlinear programming typically produce locally optimal solutions. For techniques that yield globally optimal solutions, see, e.g., Rinnooy Kan and Timmer (1989).

In the coordinate search algorithm below, $TUE(\beta_1, \beta_2, \mu_1, \mu_2)$ denotes the TUE problem in Section 4 with the pricing function based on $T^{max}(\ell) = \max\{\beta_1 + \mu_1\ell, \beta_2 + \mu_2\ell\}$. The algorithm assumes that $\beta_1 \in [0, \beta_1^{max}], \mu_1 \in [0, \mu_1^{max}], \beta_2 \in [0, \beta_2^{max}]$ and $\mu_2 \in [0, \mu_2^{max}]$.

Coordinate Search Algorithm

Step 1: Set $(\beta_1^1, \beta_2^1, \mu_1^1, \mu_2^1) = (0,0,0,0)$ and $m = 1$.

Step 2: Let β_1^{m+1} solves the following problem:

$$\max\left\{\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}: 0 \le \beta_1 \le \beta_1^{max}, (\boldsymbol{v}, \boldsymbol{d}) \in V^f, (\boldsymbol{v}, \boldsymbol{d}) \text{ solves } TUE(\beta_1, \beta_2^m, \mu_1^m, \mu_2^m)\right\} \tag{A.5.2}$$

Step 3: Let μ_1^{m+1} solves the following problem:

$$\max\left\{\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}: 0 \le \mu_1 \le \mu_1^{max}, (\boldsymbol{v}, \boldsymbol{d}) \in \boldsymbol{V^f}, (\boldsymbol{v}, \boldsymbol{d}) \text{ solves } TUE(\beta_1^{m+1}, \beta_2^m, \mu_1, \mu_2^m)\right\} \tag{A.5.3}$$

Step 4: Let β_2^{m+1} solves the following problem:

$$\max\left\{\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}: 0 \le \beta_2 \le \beta_2^{max}, (\boldsymbol{v}, \boldsymbol{d}) \in \boldsymbol{V^f}, (\boldsymbol{v}, \boldsymbol{d}) \text{ solves } TUE(\beta_1^{m+1}, \beta_2, \mu_1^{m+1}, \mu_2^m)\right\} \tag{A.5.4}$$

Step 5: Let μ_2^{m+1} solves the following problem:

$$\max\left\{\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}: 0 \le \mu_2 \le \mu_2^{max}, (\boldsymbol{v},\boldsymbol{d}) \in \boldsymbol{V}^f, (\boldsymbol{v},\boldsymbol{d}) \text{ solves } TUE(\beta_1^{m+1},\beta_2^{m+1},\mu_1^{m+1},\mu_2)\right\} \quad \text{(A.5.5)}$$

Step 6: If $\|(\beta_1^{m+1},\beta_2^{m+1},\mu_1^{m+1},\mu_2^{m+1}) - (\beta_1^m,\beta_2^m,\mu_1^m,\mu_2^m)\| \le \epsilon$, stop and $(\beta_1^{m+1},\beta_2^{m+1},\mu_1^{m+1},\mu_2^{m+1})$ solves NLT approximately. Otherwise, set $m = m + 1$ and return to Step 2.

In Step 1, it is also possible to use other values for $(\beta_1^1,\beta_2^1,\mu_1^1,\mu_2^1)$. The problems in Steps 2 – 5 essentially have only one decision variable, i.e., they can be viewed as line search problems and there are many line search algorithms in the literature (see, e.g., Bazaraa et al., 2006), all of which guarantee a globally optimal solution under some assumptions. In our implementation below, we solve, e.g., the $TUE(\beta_1,\beta_2^m,\mu_1^m,\mu_2^m)$ problem in Step 2 by SD to obtain UE flow-demand pairs at 20 equally spaced β_1-values in the interval $[0,\beta_1^{max}]$ and choose one whose UE flow-demand pair $(\boldsymbol{v},\boldsymbol{d})$ yields the best social benefit, i.e., $\sum_{k\in K}\int_0^{d_k} D_k^{-1}(\chi)d\chi - s(\boldsymbol{v})^T\boldsymbol{v}$, as the solution to the problem in Step 2. The procedures for Steps 3 – 5 are similar. The order in which to optimize the pricing parameters in Steps 2 – 5 is heuristic. Other orderings are possible and may lead to a faster convergence. In Step 6, the algorithm terminates when the change between two consecutive solutions is small.

A.6 Frank-Wolfe Algorithm

This section presents a modification of the Frank-Wolfe algorithm for solving the tolled UE problem with two-part pricing or $TUE2$. For linearly constrained convex programs, the Frank-Wolfe algorithm begins with an initial feasible solution, finds an improving feasible direction by solving a linear program that approximates the original problem, and performs a line search along the direction found to obtain an improved solution. In theory, the algorithm repeats these steps until it finds a feasible solution for which no improving feasible direction exists. When applied to the toll-free UE problems in the literature, finding an improving feasible direction reduces to solving a shortest path problem for each OD pair. The similar is true when applied to $TUE2$. Instead of one, the algorithm below solves two shortest path problems for each OD pair, one to obtain a path using the tolling area and the other to find one that bypasses it instead.

The algorithm below applies the Frank-Wolfe algorithm to $TUE2$ with the assumption that $D_k^{-1}(0) = M_k < \infty$, i.e., M_k is the maximum demand for OD pair k.

Frank-Wolfe Algorithm for TUE2

Step 1: Let $(\boldsymbol{x}^1,\boldsymbol{y}^1,\boldsymbol{\sigma}^1,\boldsymbol{\eta}^1) \in V^x$ and set $m = 1$.

Step 2: Let $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}},\hat{\boldsymbol{\sigma}},\hat{\boldsymbol{\eta}})$ solve the following (sub)problem:

$$
\begin{aligned}
SUB3^m\text{:}\quad \min\quad & \sum_{a\in\Omega_1}(s_a(v_a^m)+\mu_1 l_a)x_a^k+\sum_{a\in\Omega_2}s_a(v_a^m)(x_a^k+y_a^k)+\sum_{k\in K}\beta\sigma_k-\sum_{k\in K}D_k^{-1}(d_k^m)(\sigma_k+\eta_k)\\
\text{s.t.}\quad & A_1\boldsymbol{x}_1^k+A_2\boldsymbol{x}_2^k=\sigma_k\boldsymbol{E}_k,\forall k\in K\\
& A_2\boldsymbol{y}_2^k=\eta_k\boldsymbol{E}_k,\forall k\in K\\
& \sigma_k+\eta_k\le M_k,\forall k\in K\\
& x_a^k,y_a^k,\sigma_k,\eta_k\ge 0,\forall k\in K,a\in\Omega.
\end{aligned}
\tag{A.6.1}
$$

Step 3: If the following holds, stop and $(\boldsymbol{x}^m,\boldsymbol{y}^m,\boldsymbol{\sigma}^m,\boldsymbol{\eta}^m)$ is optimal. Otherwise, go to Step 4.

$$
\begin{aligned}
\sum_{a\in\Omega^1}(s_a(\boldsymbol{v}^m)+\mu_1 l_a)(\hat{v}_a-v_a^m)+\sum_{a\in\Omega^2}s_a(\boldsymbol{v}^m)(\hat{v}_a-v_a^m)+\sum_{k\in K}\beta(\hat{\sigma}_k-\sigma_k^m)\\
-\boldsymbol{D}^{-1}(\boldsymbol{d}^m)^T(\widehat{\boldsymbol{d}}-\boldsymbol{d}^m)\ge 0
\end{aligned}
\tag{A.6.2}
$$

where $\boldsymbol{d}^m=\boldsymbol{\sigma}^m+\boldsymbol{\eta}^m$ and $\widehat{\boldsymbol{d}}=\widehat{\boldsymbol{\sigma}}+\widehat{\boldsymbol{\eta}}$.

Step 4: Let $\lambda^m\in[0,1]$ solve the following one-dimensional problem:

$$
\begin{aligned}
\min_{\lambda\in[0,1]}\sum_{a\in\Omega^1}\int_0^{\lambda\hat{v}_a+(1-\lambda)v_a^m}(s_a(\chi)+\mu_1 l_a)d\chi+\sum_{a\in\Omega^2}\int_0^{\lambda\hat{v}_a+(1-\lambda)v_a^m}s_a(\chi)d\chi\\
+\beta\sum_{k\in K}\lambda\hat{\sigma}_k+(1-\lambda)\sigma_k^m-\sum_{k\in K}\int_0^{\lambda\hat{d}_k+(1-\lambda)d_k^m}D_k^{-1}(\chi)d\chi
\end{aligned}
\tag{A.6.3}
$$

Set $(\boldsymbol{x}^{m+1},\boldsymbol{y}^{m+1},\boldsymbol{\sigma}^{m+1},\boldsymbol{\eta}^{m+1})=\lambda^m(\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{\sigma}},\widehat{\boldsymbol{\eta}})+(1-\lambda^m)(\boldsymbol{x}^m,\boldsymbol{y}^m,\boldsymbol{\sigma}^m,\boldsymbol{\eta}^m)$ and $m=m+1$. Return to Step 2.

It is possible to let $(\boldsymbol{x}^1,\boldsymbol{y}^1,\boldsymbol{\sigma}^1,\boldsymbol{\eta}^1)=(0,0,0,0)$ in Step 1. Problem $SUB3^m$ in Step 2 is a linear program that approximates $TUE2$ around the current solution $(\boldsymbol{x}^m,\boldsymbol{y}^m,\boldsymbol{\sigma}^m,\boldsymbol{\eta}^m)$. The first two constraints in $SUB3^m$ balance the flows at each node for paths that use and do not use the tolling area, respectively. The third set of constraints ensures that the demands for toll and toll-free routes do not exceed the maximum for each OD pair. Equivalently, $SUB3^m$ can be written as follows:

$$
\begin{aligned}
SUB3a^m\text{:}\quad min\quad & \sum_{k\in K}\pi_k^1(\sigma_k)-D_k^{-1}(d_k^m)\sigma_k+\sum_{k\in K}\pi_k^2(\eta_k)-D_k^{-1}(d_k^m)\eta_k\\
s.t.\quad & \sigma_k+\eta_k\le M_k,\forall k\in K\\
& \sigma_k,\eta_k\ge 0,\forall k\in K
\end{aligned}
\tag{A.6.4}
$$

where, for each $k\in K$,

$$
\begin{aligned}
\pi_k^1(\sigma_k)&=\beta\sigma_k+\min\left\{\sum_{a\in\Omega}(s_a(v_a^m)+\mu_1 l_a)x_a^k : A\boldsymbol{x}^k=\sigma_k\boldsymbol{E}_k,x^k\ge 0\right\},\\
\pi_k^2(\eta_k)&=\min\left\{\sum_{a\in\Omega^2}s_a(v_a^m)y_a^k : A_2\boldsymbol{y}^k=\eta_k\boldsymbol{E}_k,z^k\ge 0\right\}.
\end{aligned}
\tag{A.6.5}
$$

The minimization problems in the definition of $\pi_k^1(\sigma_k)$ and $\pi_k^2(\eta_k)$ are minimum cost flow problems (see, e.g., Ahuja et al., 1993). Because there is no capacity constraint on any link, these minimizations correspond to sending σ_k and η_k along the least-cost path using and not using the tolling area, respectively.

To solve $SUB3a^m$, evaluate $\pi_k^1(M_k)$ and $\pi_k^2(M_k)$, i.e., solve two shortest path problems, one using the full network and the other bypassing the tolling area, and send M_k units of flows along each route. Then, the solution to $SUB3a^m$ is, for each k,

$$(\hat{\sigma}_k, \hat{\eta}_k) = \begin{cases} (0,0), & \text{if } \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k \geq 0 \\ (M_k, 0), & \text{if } \pi_k^1(M_k) \leq \pi_k^2(M_k) \text{ \& } \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k < 0 \\ (0, M_k), & \text{if } \pi_k^2(M_k) < \pi_k^1(M_k) \text{ \& } \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k < 0 \end{cases} \quad \text{(A.6.6)}$$

Then, the corresponding the optimal solution to $SUB3^m$ is

$$(\hat{x}^k, \hat{y}^k, \hat{\sigma}_k, \hat{\eta}_k) = \begin{cases} (0,0,0,0), & \text{if } \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k \geq 0 \\ (x^k, 0, M_k, 0), & \text{if } \pi_k^1(M_k) \leq \pi_k^2(M_k) \text{ \& } \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k < 0 \\ (0, y^k, 0, M_k), & \text{if } \pi_k^2(M_k) < \pi_k^1(M_k) \text{ \& } \min\{\pi_k^1(M_k), \pi_k^2(M_k)\} - D_k^{-1}(d_k^m)M_k < 0 \end{cases} \quad \text{(A.6.7)}$$

where x^k and y^k are optimal solutions to the minimization problems in $\pi_k^1(M_k)$ and $\pi_k^2(M_k)$, respectively.

To obtain a more efficient algorithm, it is also possible to modify or extend the above algorithm via simplicial decomposition (see, e.g., Lawphongpanich and Hearn, 1984, Hearn et al., 1987, and Patriksson, 1994).